THE MAPPING CLASS GROUP CANNOT BE REALIZED BY HOMEOMORPHISMS

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Abstract. Let M be a closed surface. By $\mathrm{Homeo}(M)$ we denote the group of orientation preserving homeomorphisms of M and let $\mathcal{MC}(M)$ denote the Mapping class group. In this paper we complete the proof of the conjecture of Thurston that says that for any closed surface M of genus $\mathbf{g} \geq 2$, there is no homomorphic section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$ of the standard projection map $\mathcal{P} : \mathrm{Homeo}(M) \to \mathcal{MC}(M)$.

1. Introduction

Let M be a closed surface of genus $\mathbf{g} \geq 2$. If $\widetilde{f} \in \mathrm{Homeo}(M)$ is a homeomorphism, then $[\widetilde{f}] = f \in \mathcal{MC}(M)$ denotes the corresponding homotopy class. Denote by $\mathcal{P} : \mathrm{Homeo}(M) \to \mathcal{MC}(M)$, the projection from the group of orientation preserving homeomorphisms $\mathrm{Homeo}(M)$ of M, onto the mapping class group $\mathcal{MC}(M)$, that is for $\widetilde{f} \in \mathrm{Homeo}(M)$, set $\mathcal{P}(\widetilde{f}) = [\widetilde{f}]$. One stimulating question is whether there exists a homomorphism $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$, so that $\mathcal{P} \circ \mathcal{E}$ is the identity mapping on $\mathcal{MC}(M)$. Such a homomorphism represents a homomorphic section (from now we just say section) of the projection $\mathcal{P}$. The mapping class group of the torus can be represented by homeomorphisms, namely the corresponding section $\mathcal{E}$ exists. In fact, it can be represented as the group of affine transformations $\mathbf{SL}_2(\mathbf{Z})$.

Morita [10] showed that there is no such section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Diff}(M)$, when $\mathbf{g} > 4$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms of M. Markovic [7] showed that a section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$ does not exists when $\mathbf{g} > 5$. Very recently Franks and Handel [4] showed that $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Diff}(M)$ does not exists when $\mathbf{g} \geq 3$. In [1] Cantat and Cerveau showed that there is no section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Diff}^{\omega}(M)$ for any $\mathbf{g} \geq 2$, where $\mathrm{Diff}^{\omega}(M)$ is the group of real analytic diffeomorphisms of M. In fact in [4] and [1] it is shown that such sections do not exist when $\mathcal{MC}(M)$ is replaced by a finite index subgroup of $\mathcal{MC}(M)$. In this paper we settle the general case, by showing that such $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$ does not exists, where M is a closed surface of any genus $\mathbf{g} \geq 2$. This of course settles the case of diffeomorphisms in the genus two case as well. The main result of this paper is the following theorem.

Theorem 1.1. *Let M be a closed surface of genus $\mathbf{g} \geq 2$. Let $\mathcal{P} : \mathrm{Homeo}(M) \to \mathcal{MC}(M)$ be the projection. Then there is no homomorphism $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$, so that $\mathcal{P} \circ \mathcal{E}$ is the identity mapping on $\mathcal{MC}(M)$.*

The proof of this theorem is based on analysing certain Artin type relations in $\mathcal{MC}(M)$, proved by Farb-Margalit in [2], and eventually obtaining a contradiction

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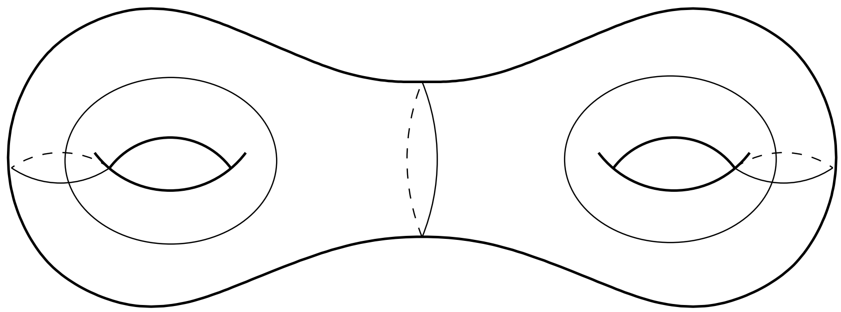

Figure 1. The genus two case.

with the existence of a homomorphic section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$. Our proof does not distinguish between surfaces of different genus, except that we treat surfaces of even and odd genus in a slightly different manner (the differences are cosmetic). We use techniques from [7] but ultimately we need several new ideas and technical gadgets to prove this theorem.

We state these important relations in $\mathcal{MC}(M)$ and recall the notion of upper semi-continuous decompositions and the minimal decomposition for subgroups of $\mathrm{Homeo}(M)$ in Section 2. In Section 3 we introduce the notion of the twist number that is associated to a homeomorphism of an annulus, and prove the main preliminary results about dynamics of homeomorphisms actions on annuli. In Section 4 we assume the existence of a homomorphic section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$ and construct the minimal annulus (this is a certain topological annulus in M where the argument takes place). In Section 5 we prove Theorem 1.1. The Artin type relations from Section 2 will be used toward the end of Section 5.

2. Important relations in $\mathcal{MC}(M)$ and the minimal decomposition

2.1. **Relations in the mapping class group.** Recall that for a simple closed curve α by $t_\alpha \in \mathcal{MC}(M)$ we denote the twist about α. Given two simple closed curves α and β on M, let $([\alpha],[\beta])$ denote the geometric intersection number between their homotopy classes.

First consider the case when M is a closed surface of even genus $\mathbf{g} \geq 2$. Then there exists a separating simple closed curve γ on M such that $M \setminus \gamma$ has two components each homeomorphic to a closed surface of genus $\mathbf{g}/2$ minus a disk (see Figure 1 and Figure 2). Let $e \in \mathcal{MC}(M)$ be an involution (that is $e^2 = id$) which interchanges the two components of $M \setminus \gamma$ and such that $e([\gamma]) = [\gamma]$. There are many such involutions and we fix one of them once and for all.

Let $\alpha_1, \dots, \alpha_g$ be a *chain* of simple closed curves on the left-hand side component of $M \setminus \{\gamma\}$, namely $([\alpha_i],[\alpha_{i+1}]) = 1$ for $i = 1, \dots, g$, and $([\alpha_i],[\alpha_j]) = 0$ for $i,j = 1, \dots, g$ with $|i-j| \geq 2$. Let β_i be curves such that $e([\alpha_i]) = [\beta_i]$, for every i. Then $\beta_1, \dots, \beta_g$ is a chain of simple closed curves on the right-hand side component of $M \setminus \gamma$.

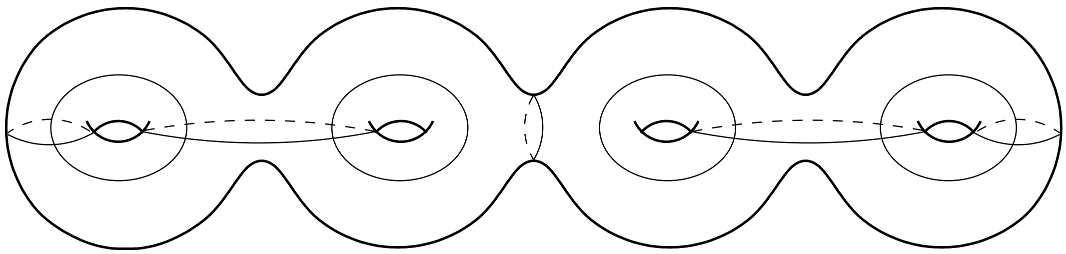

FIGURE 2. The genus four case.

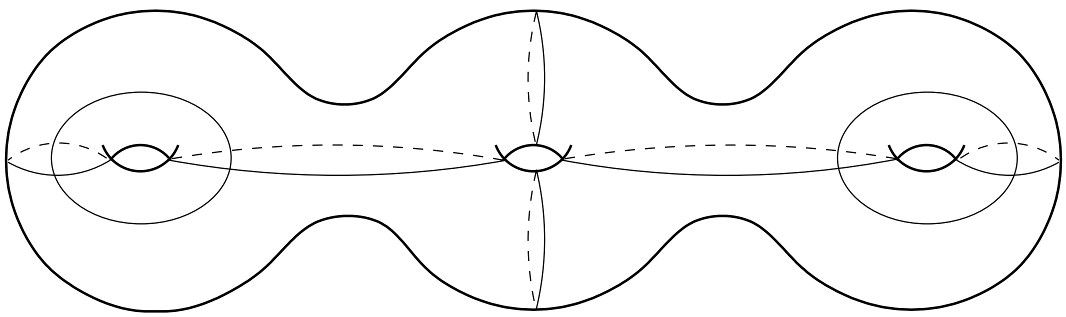

FIGURE 3. The genus three case.

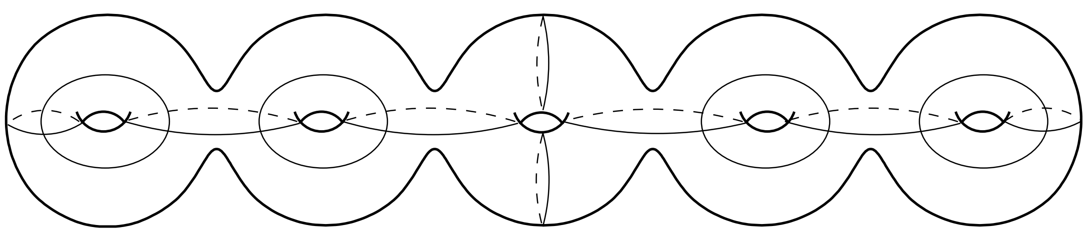

FIGURE 4. The genus five case.

The following Artin type relations are derived in the survey paper by Farb-Margalit (see [2])

$$(t_{\alpha_1} \circ \cdots \circ t_{\alpha_g})^{2g+2} = t_\gamma = (t_{\beta_1} \circ \cdots \circ t_{\beta_g})^{2g+2}. \tag{1}$$

By $\mathbf{a}_1$ we denote the set of curves α_{2i-1}, $i = 1, \ldots, g/2$, and by $\mathbf{a}_2$ we denote the set of curves α_{2i}, $i = 1, \ldots, g/2$. Similarly, we denote by $\mathbf{b}_1$ the set of curves β_{2i-1}, $i = 1, 2, \ldots, g/2$, and denote by $\mathbf{b}_2$ the set of curves β_{2i}, $i = 1, 2, \ldots, g/2$. We have $e([\mathbf{a}_j]) = [\mathbf{b}_j]$, where $j = 1, 2$.

Next consider the case when M is a closed surface of odd genus $\mathbf{g} \geq 3$. Let γ and γ_1 be two simple closed non-intersecting curves on M such that $M \setminus (\gamma \cup \gamma_1)$ has two components that are both homeomorphic to a twice holed surface of genus $(g-1)/2$. Let $e \in \mathcal{MC}(M)$ be an involution which interchanges the two components of $M \setminus (\gamma \cup \gamma_1)$, and such that $e([\gamma]) = [\gamma]$, and $e([\gamma_1]) = [\gamma_1]$ (there are many such involutions and we fix one of them from now on). Similarly as above (see Figure 3 and Figure 4) let $\alpha_1, \ldots, \alpha_g$ be a chain of simple closed curves on the left-hand side component of $M \setminus (\gamma \cup \gamma_1)$. Let β_i be such that $[\beta_i] = e([\alpha_i])$. Then $\beta_1, \ldots, \beta_g$ is a chain of simple closed curves on the right-hand side component of $M \setminus (\gamma \cup \gamma_1)$. We have the relations (see [2])

$$(t_{\alpha_1} \circ \cdots \circ t_{\alpha_g})^{g+1} = t_\gamma t_{\gamma_1} = (t_{\beta_1} \circ \cdots \circ t_{\beta_g})^{g+1}. \tag{2}$$

We denote by $\mathbf{a}_1$ the set of curves α_{2i-1} for $i = 1, \ldots, (g+1)/2$. By $\mathbf{a}_2$ the set of curves α_{2i} for $i = 1, \ldots, (g-1)/2$ together with the curve γ_1 as in Figure 3 and Figure 4. Similarly, we denote by $\mathbf{b}_1$ the set of curves β_{2i-1} for $i = 1, 2, \ldots, (g+1)/2$, and denote by $\mathbf{b}_2$ the set of curves β_{2i} for $i = 1, 2, \ldots, (g-1)/2$ together with the curve γ_1. Then $e([\mathbf{a}_i]) = [\mathbf{b}_i]$.

The relations (1) and (2) are the nontrivial relations we use in the paper. Beside this we will frequently use the following. If α and β are simple closed curves such that $([\alpha], [\beta]) = 0$ then t_α and t_β commute. Also, if $f \in \mathcal{MC}(M)$ and $[\beta] = f([\alpha])$ then $f^{-1} \circ t_\beta \circ f = t_\alpha$.

The strategy of our proof is that under the assumption that a section $\mathcal{E}$: $\mathcal{MC}(M) \to \mathrm{Homeo}(M)$ exists, obtain a contradiction with the relations introduced above.

2.2. The minimal decomposition.

Let M be a closed surface of genus $\mathbf{g} \geq 2$. First we recall several definition and results from [7] that are need in this paper.

Definition 2.1. *Let $\mathbf{S}$ be a collection of closed, connected subsets of M. We say that $\mathbf{S}$ is an upper semi-continuous decomposition of M if the following holds:*

1. *If $S_1, S_2 \in \mathbf{S}$, then $S_1 \cap S_2 = \emptyset$.*
2. *If $S \in \mathbf{S}$, then the set $M \setminus S$ does not contain a connected component that is simply connected.*
3. *We have*

$$M = \bigcup_{S \in \mathbf{S}} S.$$

4. *If $S_n \in \mathbf{S}$, $n \in \mathbf{N}$, is a sequence that has the Hausdorff limit S_0, then there exists $S \in \mathbf{S}$ such that $S_0 \subset S$.*

From now on by $\mathbf{S}$ we always denote an upper semi-continuous decomposition of M. Recall that a component $S \in \mathbf{S}$ is said to be acyclic if there is a simply connected open set $U \subset M$ such that $S \subset U$ and $U \setminus S$ is homeomorphic to an annulus. For every point $p \in M$, there exists a unique component in $\mathbf{S}$ that contains p. We denote this component by $S_p \in \mathbf{S}$. The set of all points $p \in M$ such that the corresponding S_p is acyclic is denoted by $M_{\mathbf{S}}$. The set $M_{\mathbf{S}}$ is open and every connected component of $M_{\mathbf{S}}$ is a proper subsurface of M, which represents the interior of a compact subsurface of M with finitely many ends.

Definition 2.2. *Let* $\mathbf{S}$ *be an upper semi-continuous decomposition of* M. *Let* G *be a subgroup of* $\mathrm{Homeo}(M)$. *We say that* $\mathbf{S}$ *is admissible for the group* G *if the following holds.*

(1) *Each* $\widehat{f} \in G$ *preserves setwise every component of* $\mathbf{S}$.
(2) *Let* $S \in \mathbf{S}$. *Then every point, in every frontier component of the surface* $M \setminus S$, *is a limit of points from* $M \setminus S$ *that belong to acyclic components of* $\mathbf{S}$ *(note that not every point of* S *need to be in a frontier component of the subsurface* $M \setminus S$*).*

If G *is a cyclic group generated by a homeomorphism* $\widehat{f} : M \to M$ *we say that* $\mathbf{S}$ *is an admissible decomposition for* $\widehat{f}$.

For a generic homeomorphism $\widehat{f} : M \to M$, the only admissible decomposition is the trivial one, which is the one that contains only one set, namely M itself.

Definition 2.3. *An admissible decomposition* $\mathbf{S}$ *for a group* G *will be called the minimal decomposition for* G *if* $\mathbf{S}$ *is contained in every admissible decomposition for* G.

We have [7]

Theorem 2.1. *Every group* $G < \mathrm{Homeo}(M)$ *has the unique minimal decomposition* $\mathbf{S}(G)$. *That is, if* $\mathbf{S}$ *is an admissible decomposition for* G *then for every* $p \in M$ *we have* $S_p(G) \subset S_p$, *where* $S_p(G) \in \mathbf{S}(G)$ *and* $S_p \in \mathbf{S}$.

Assuming that the section $\mathcal{E} : \mathcal{MC}(M) \to \mathrm{Homeo}(M)$ exists, we have the following lemma.

Lemma 2.1. *Let* $\mathbf{a}$ *denote a set of simple, closed, and mutually disjoint curves on* M, *such that no two curves are homotopic and no curve is homotopically trivial. Let* $G < \mathrm{Homeo}(M)$ *be the group generated by* $\mathcal{E}(t_\alpha)$, *where* $\alpha \in \mathbf{a}$, *and let* $\mathbf{S}$ *denote the minimal decomposition for* G. *Suppose that* R *is a connected component of the open set obtained by removing the collection of curves* $\mathbf{a}$ *from* M. *If* R *has the negative Euler characteristic then there exists a unique component of* $M_{\mathbf{S}}$ *homotopic to* R.

Proof. The set obtained by removing the collection of curves $\mathbf{a}$ from M, is a finite and disjoint union of surfaces with boundary. That is, each such surface is obtained by removing disjoint discs from closed surface. Assume that R is one of these surface with boundary, and assume that R has the negative Euler characteristic. We will show that there exists a component of $M_{\mathbf{S}}$ that is homotopic to R.

Let $T \subset R$, be a subsurface of R, that is either homeomorphic to a torus minus a disc, or to a sphere minus four disc. Since there exists an Anosov diffeomorphism on such T it follows from Theorem 4.1 in [7] that we can find a surface $T_1 \subset M_{\mathbf{S}}$ that is homotopic to T.

Remark. In [7] this was proved in the case when T is a torus minus a disc, but the proof is the same when T is a sphere with four holes. This lemma can be proved in a similar way by using the results from [3].

Let α be a simple closed curve in R, that is not homotopic to an end of R, and that is not homotopically trivial. Since R has the negative Euler characteristic we can find a surface $T \subset R$, that is either homeomorphic to a torus minus a disc, or to a sphere minus four disc, and such that T contains a simple closed curve α_1 that is homotopic to α. This shows that for every such curve $\alpha \subset R$ there exists a curve α_1 that is homotopic to α, and that belongs to $M_{\mathbf{S}}$. This implies that there exists a component of $M_{\mathbf{S}}$ that contains a curve homotopic to any simple closed curve on R. Denote this component by M_1. We see that M_1 is homotopic to R. The uniqueness is obvious. □

Recall the following definition.

Definition 2.4. *Let $K \subset M$ be a closed and connected set. We say that K is a triode if there exists a connected closed set $K_1 \subset K$, such that $K \setminus K_1$ has at least three connected components.*

The Moore's triode theorem says that any open subset of M can contain at most countably many disjoint triodes (see [8]), [9]).

Lemma 2.2. *Let F and G be two groups of homeomorphisms of M such that $\widetilde{f}$ commutes with $\widetilde{g}$ for every $\widetilde{f} \in F$ and $\widetilde{g} \in G$. Denote by $\mathbf{S}(F)$ and $\mathbf{S}(G)$ the minimal decompositions that correspond to the groups F and G respectively. Let Γ be the group generated by the elements from F and G and let $\mathbf{S}(\Gamma)$ be the corresponding minimal decomposition. By $M_{\mathbf{S}(\Gamma)}$ we denote the set of all points that are contained in acyclic components of $\mathbf{S}(\Gamma)$. Let $p \in M_{\mathbf{S}(\Gamma)}$ and assume that p does not belong to the interior of $S_p(\Gamma) \in \mathbf{S}(\Gamma)$. Then at least one of the following two statements holds*

- *$\widetilde{f}(S_p(G)) = S_p(G)$, for every $\widetilde{f} \in F$.*
- *$\widetilde{g}(S_p(F)) = S_p(F)$, for every $\widetilde{g} \in G$.*

(recall that S_p denotes the component from $\mathbf{S}$ that contains the point p).

Proof. Assume that $M_{\mathbf{S}(\Gamma)}$ is non-empty (otherwise the lemma is trivial). Then by the minimality we have $M_{\mathbf{S}(\Gamma)} \subset (M_{\mathbf{S}(F)} \cap M_{\mathbf{S}(G)})$. Let $p \in M_{\mathbf{S}(\Gamma)}$ and assume that

$$\widetilde{f}(S_p(G)) \neq S_p(G), \quad \text{and} \quad \widetilde{g}(S_p(F)) \neq S_p(F), \tag{3}$$

for some $\widetilde{f} \in F$ and $\widetilde{g} \in G$. Since the groups F and G commute we have that $\widetilde{f}$ respects the minimal decomposition $\mathbf{S}(G)$ and that $\widetilde{g}$ respects the minimal decomposition $\mathbf{S}(F)$. This implies that no component $S_q(G) \in \mathbf{S}(G)$ is a subset of $S_p(F)$ (if $S_q(G) \subset S_p(F)$ then $\widetilde{g}(S_p(F)) = S_p(F)$). Similarly we have that $S_p(F)$ is not a subset of $S_p(G)$. Therefore we can find a point $p_1 \in S_p(F)$ such that p_1 does not belong to $S_p(G)$. We have that $S_p(G)$ and $S_{p_1}(G)$ are different components from $\mathbf{S}(G)$ and therefore they are mutually disjoint.

We already observed that $S_p(F)$ is not a subset of any $S_q(G)$. We show that $S_p(F)$ is not a subset of $S_p(G) \cup S_{p_1}(G)$. Consider the set $X = S_p(F) \setminus S_p(G)$. Then X is a relatively open subset of $S_p(F)$. If $S_p(F) \subset \big(S_p(G) \cup S_{p_1}(G)\big)$, and since $S_p(G) \cap S_{p_1}(G) = \emptyset$ we have that $X = S_p(F) \cap S_{p_1}(G)$. This implies that X is both relatively open and closed in $S_p(F)$ which shows that $X = S_p(F)$. This is a contradiction so we have that $S_p(F)$ is not a subset of $S_p(G) \cup S_{p_1}(G)$.

Therefore there exists $p_2 \in S_p(F)$ such that $S_{p_2}(G)$ is disjoint from $S_p(G)$ and $S_{p_1}(G)$. Consider now the component $S_p(\Gamma) \in \mathbf{S}(\Gamma)$. By the minimality we have that

$$Y = S_p(F) \cup S_p(G) \cup S_{p_1}(G) \cup S_{p_2}(G) \subset S_p(\Gamma).$$

On the other hand Y is a connected closed set and $Y \setminus S_p(F)$ contains at least three connected components. This shows that Y is a triode and we conclude that if for some $p \in M_{\mathbf{S}(\Gamma)}$ we have that (3) holds then $S_p(\Gamma)$ contains a triode. By the Moore's theorem there could be at most countably many such components in $\mathbf{S}(\Gamma)$. On the other hand the set of points $p \in M_{\mathbf{S}(\Gamma)}$ such that at least one of the two conditions from the statement of this lemma holds is relatively closed in $M_{\mathbf{S}(\Gamma)}$. In particular if $p_0 \in S_p(\Gamma) \in M_{\mathbf{S}(\Gamma)}$ is such that p_0 does not belong to the interior of $S_p(\Gamma)$, then there exists a sequence of points $p_n \in M_{\mathbf{S}(\Gamma)}$ such that at least one of the two conditions holds at p_n and $p_n \to p_0$ (this follows from the definition of admissible decompositions). Then at least one of the two conditions holds at p_0. This proves the lemma.

$\square$

3. The twist number and the analysis on the strip

3.1. **The Translation number.** Let $\varphi : \mathbf{R} \to \mathbf{R}$, be a homeomorphism (orientation preserving) that commutes with the translation $T(x) = x + 1$. By the classical result of Poincare, the limit

$$\rho(\varphi) = \lim_{n\to\infty} \frac{1}{n}\varphi^n(x),$$

exists, and it does not depend on $x \in \mathbf{R}$. The number $\rho(\varphi)$ is called the *translation number* of φ.

Assume that φ has a fixed point, that is $\varphi(x_0) = x_0$ for some $x_0 \in \mathbf{R}$. Then $\varphi(x_0+k) = x_0+k$ for every $k \in \mathbf{Z}$. Therefore for every $x \in \mathbf{R}$, we have $|\varphi^n(x)-x| < 1$ for every $n \in \mathbf{N}$, so in this case $\rho(\varphi) = 0$. If φ does not have a fixed point then either for every $x \in \mathbf{R}$ we have $\varphi(x) > x$, or for every $x \in \mathbf{R}$ we have $\varphi(x) < x$. Suppose that $\varphi(x) > x$, $x \in \mathbf{R}$. Since φ commutes with the translation $T(x)$, by the compactness there exists $q > 0$, such that $\varphi(x) > x + q$ for every $x \in \mathbf{R}$. We have $\varphi^n(x) > x + nq$, which shows that $\rho(\varphi) > 0$. Similarly, if $\varphi(x) < x$, $x \in \mathbf{R}$, then $\rho(\varphi) < 0$. We conclude that the translation number is zero if and only if φ has a fixed point.

Remark. Since φ commutes with $T(x)$ we have that φ is a lift of the circle homeomorphism φ_1. The rotation number of φ_1 is defined to be the translation number of φ modulo 1. It is not true that the rotation number of φ_1 is equal to zero if and only if φ_1 has a fixed point on the circle. However, the classical result says that rotation number of φ_1 is a rational number if and only if some power of φ_1 has a fixed point on the circle.

Note that $\rho(\varphi^m) = m\rho(\varphi)$, and $\rho(T^m \circ \varphi) = \rho(\varphi \circ T^m) = \rho(\varphi) + m$, for any $m \in \mathbf{Z}$.

Proposition 3.1. *Let $\varphi : \mathbf{R} \to \mathbf{R}$, be a homeomorphism (orientation preserving) that commutes with the translation $T(x)$. Then for every $x \in \mathbf{R}$, and for every $n \in \mathbf{N}$, we have*

$$|(\varphi^n(x) - x) - n\rho(\varphi)| < 3.$$

Remark. This proposition shows that given a compact set $A \subset \mathbf{R}$, the sequence $\frac{1}{n}\varphi^n(x)$ converges uniformly to $\rho(\varphi)$, for every $x \in A$, regardless of the choice of the homeomorphism φ.

Proof. Assume that for some $x_0 \in \mathbf{R}$, and some $n_0 \in \mathbf{N}$, we have

$$|(\varphi^{n_0}(x) - x) - n_0\rho(\varphi)| \geq 3.$$

Then either

$$\varphi^{n_0}(x_0) - x_0 > n_0\rho(\varphi) + 3,$$

or

$$\varphi^{n_0}(x_0) - x_0 < n_0\rho(\varphi) - 3.$$

Consider the first case. Note that for every $x_1, x_2 \in \mathbf{R}$, with $|x_1 - x_2| < 1$, we have $|\varphi^n(x_1) - \varphi^n(x_2)| < 1$, for every $n \in \mathbf{N}$. This implies that for every $x \in \mathbf{R}$, we have

$$\varphi^{n_0}(x) - x > (\varphi^{n_0}(x_0) - x_0) - 2 \geq n_0\rho(\varphi) + 3 - 2 = n_0\rho(\varphi) + 1.$$

This yields that for every $j \in \mathbf{N}$, by setting $x = \varphi^{(j-1)n_0}(x_0)$, the above inequality yields

$$\varphi^{jn_0}(x_0) - \varphi^{(j-1)n_0}(x_0) > n_0\rho(\varphi) + 1.$$

Let $k \in \mathbf{N}$, and $1 \leq j \leq k$. We sum up all the above inequalities for $1 \leq j \leq k$, and get

$$\varphi^{kn_0}(x_0) - x_0 > kn_0\rho(\varphi) + k.$$

Letting $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \frac{1}{kn_0}(\varphi^{kn_0}(x_0) - x_0) \geq \rho(\varphi) + \frac{1}{n_0} > \rho(\varphi).$$

This is a contradiction. The second case is handled in the same way. $\square$

3.2. **The twist number of an annulus homeomorphism.** In this section z and w represent complex variables in the complex plane $\mathbf{C}$. We have $Re(z) = x$ and $Im(z) = y$, that is $z = x + iy$. Let $N(r) = \{w \in \mathbf{C} : \frac{1}{r} < |w| < r\}$, be the geometric annulus in the complex plane $\mathbf{C}$. By $P(r) = \{x + iy = z \in \mathbf{C} : |y| < \frac{\log r}{2\pi}\}$, we denote the geometric strip in $\mathbf{C}$. By $\overline{N(r)}$ and $\overline{P(r)}$, we denote the corresponding closures of $N(r)$ and $P(r)$ in the complex plane $\mathbf{C}$.

Remark. We point out that $\overline{P(r)}$ is the closure of $P(r)$ in $\mathbf{C}$, that is ∞ does not belong to $\overline{P(r)}$.

Let $\partial_0(N(r)) = \{w \in \mathbf{C} : |w| = \frac{1}{r}\}$, and $\partial_1(N(r)) = \{w \in \mathbf{C} : |w| = r\}$. Similarly, set $\partial_0(P(r)) = \{z \in \mathbf{C} : y = -i\frac{\log r}{2\pi}\}$, and $\partial_1(P(r)) = \{z \in \mathbf{C} : y = i\frac{\log r}{2\pi}\}$. The map given by $w = e^{-2\pi i z}$, is a holomorphic covering of the annulus $N(r)$ by the strip $P(r)$. Note that for every r, the covering group that acts on $P(r)$ is generated by the translation $T(z) = z + 1$.

By $\widetilde{e} : \overline{N(r)} \to \overline{N(r)}$, we always denote a conformal involution that exchanges the two boundary circles. There are exactly two such involutions and they are given by $\widetilde{e}(w) = \frac{1}{w}$, or $\widetilde{e}(w) = \frac{e^{i\pi}}{w}$. By $\widehat{e} : \overline{P(r)} \to \overline{P(r)}$ we denote a lift of $\widetilde{e}$ to $\overline{P(r)}$. For our purposes it is important to observe that every such $\widehat{e} : \overline{P(r)} \to \overline{P(r)}$ is an isometry in the Euclidean metric.

In the remainder of this section we fix $1 < r_0$ and set $N(r_0) = N$ and $P(r_0) = P$.

Definition 3.1. *Let $\widehat{f}:\overline{P}\to\overline{P}$, be a homemorphism that fixes setwise the boundary components of P, and that commutes with the translation $T(z)=z+1$. We define the twist number $\rho(\widehat{f},P)\in\mathbf{R}$ as*

$$\rho(\widehat{f},P)=\lim_{n\to\infty}\frac{1}{n}\left(Re(\widehat{f}^n(z_1))-Re(\widehat{f}^n(z_0))\right),$$

where $z_0\in\partial_0(P)$, and $z_1\in\partial_1(P)$.

Note that the restriction of $\widehat{f}$ to $\partial_0 P$ is a homeomorphism, that commutes with the translation $T(z)$. Therefore the sequence $\frac{1}{n}Re(\widehat{f}^n(z_0))$ converges to the translation number of the corresponding homeomorphism of the real line. Moreover, this limit does not depend on the choice of $z_0\in\partial_0(P)$. Similarly the sequence $\frac{1}{n}Re(\widehat{f}^n(z_1))$ converges, and this limit does not depend on the choice of $z_1\in\partial_1(P)$. This shows that $\rho(\widetilde{f},N)$ represents the difference in the translation numbers between the restriction of $\widehat{f}$ to $\partial_1(P)$, and the restriction of $\widehat{f}$ to $\partial_0(P)$.

Definition 3.2. *Let $\widetilde{f}:\overline{N}\to\overline{N}$, be a homemorphism that fixes setwise the boundary circles of N. We define the twist number $\rho(\widetilde{f},N)\in\mathbf{R}$ as follows. Let $\widehat{f}:\overline{P}\to\overline{P}$ be a lift of $\widetilde{f}$. Then $\rho(\widetilde{f},N)=\rho(\widehat{f},P)$.*

Note that the assumption that $\widetilde{f}$ setwise preserves the boundary circles of N implies that $\widehat{f}$ setwise preserves the boundary lines of P. Since any two lifts of $\widetilde{f}$ to P, differ by a translation, we see that $\rho(\widetilde{f},N)$ does not depend on the choice of the lift $\widehat{f}$. For every $m\in\mathbf{Z}$ we have $\rho(\widetilde{f}^m,N)=m\rho(\widetilde{f},N)$. Moreover, if $\widetilde{f}$ has at least one fixed point on both boundary circles then the twist number $\rho(\widetilde{f},N)$ is an integer. If $\widetilde{f}$ is homotopic to the geometric twist homeomorphism (modulo these fixed points) then $\rho(\widetilde{f},N)=1$.

Remark. If two homeomorphisms of $\overline{N}$ agree on the boundary of N, and if they are homotopic modulo the boundary, then the twists numbers agree. Moreover, our definition of the twist number of a homeomorphism of the annulus N, should not be confused with the standard definition of the rotation number for homeomorphisms of two dimensional domains (including the annulus) which very much depends on a particular homeomorphism, and not only on its homotopy class.

Let S denote a compact Riemann surface (either closed or with boundary). Let $A\subset S$ be a topological annulus. Then A has two ends. Moreover A has two frontier components $\partial_0(A)$ and $\partial_1(A)$, each corresponding to one of the ends. Although the boundary ∂A of A is the union of $\partial_0(A)$ and $\partial_1(A)$, we do not call $\partial_0(A)$ and $\partial_1(A)$ the boundary components of A because in general the sets $\partial_0(A)$ and $\partial_1(A)$ may not be disjoint. We call them frontier components of A.

Definition 3.3. *Let $A\subset S$ be a topological annulus. Let $\widetilde{f}:\overline{S}\to\overline{S}$ be a homeomorphism such that $\widetilde{f}(A)=A$, and such that $\widetilde{f}$ setwise fixes each of the two frontier components of A. Let $\Phi:N\to A$ be a conformal map and set $\widetilde{g}=\Phi^{-1}\circ\widetilde{f}\circ\Phi$.*

- *We define the twist number $\rho(\widetilde{f},A)$ to be equal to $\rho(\widetilde{g},N)$.*
- *We say that $\widetilde{f}$ has a conformal fixed point on $\partial_i(A)$, $i=0,1$, if $\widetilde{g}$ has a fixed point on $\partial_i(N)$.*

Since $\widetilde{f}$ is a homeomorphism of $\overline{A}$ it follows that $\widetilde{g}$ is a homeomorphism of $\overline{N}$, so the twist number $\rho(\widetilde{g}, N)$ is well defined. If $\widetilde{f}$ has conformal fixed points on both of its frontier components then $\rho(\widetilde{f}, A)$ is an integer.

Proposition 3.2. *Let $\widehat{f} : \overline{P} \to \overline{P}$, be a homemorphism that setwise fixes the boundary lines of P, and that commutes with the translation $T(z) = z + 1$. Then for every $z_0 \in \partial_0(P)$, $z_1 \in \partial_1(P)$, we have*

$$\Big| \big(Re(\widehat{f}^n(z_1)) - Re(\widehat{f}^n(z_0))\big) - (Re(z_1) - Re(z_0)) - n\rho(\widehat{f}, P) \Big| < 6,$$

for every $n \in \mathbf{N}$.

Proof. Let ρ_1 denote the translation number of the homeomorphism of the real line that is the restriction of $\widehat{f}$ to $\partial_1(P)$. Similarly ρ_0 denotes the translation number of the homeomorphism of the real line that is the restriction of $\widehat{f}$ to $\partial_0(P)$. Then Proposition 3.1 yields

$$\Big| \big(Re(\widehat{f}^n(z_1)) - Re(z_1)\big) - n\rho_1 \Big| < 3,$$

and

$$\Big| Re(\widehat{f}^n(z_0)) - Re(z_0) - n\rho_0 \Big| < 3.$$

Since $\rho(\widehat{f}, P) = \rho_1 - \rho_0$, by subtracting the second inequality from the first we obtain

$$\Big| \big(Re(\widehat{f}^n(z_1)) - Re(\widehat{f}^n(z_0))\big) - (Re(z_1) - Re(z_0)) - n\rho(\widehat{f}, P) \Big| < 6.$$

□

For two smooth oriented arcs h_1 and h_2 such that the set $h_1 \cap h_2$ has finitely many points, by $\iota(h_1, h_2)$ we denote their algebraic intersection number. Let l be an oriented Jordan arc in N that connects the two boundary circles $\partial_0(N)$ and $\partial_1(N)$. It is understood that such l has one endpoint on each boundary circle, and the relative interior of the arc l is contained in N. The homotopy class (modulo the endpoints) of l is the collection of all such arcs that have the same endpoints as l and are homotopic to l in N, modulo the endpoints, and that are endowed with the orientation such that the endpoints have the same order with respect to this orientation. This class of arcs is denoted by $[l]$. Given two such arcs l_1 and l_2 (that may not have the same endpoints), by $\iota([l_1], [l_2]) \in \mathbf{Z}$ we denote the algebraic intersection number between the two homotopy classes. This is well defined, since we can find smooth representatives $h_j \in [l_j]$, $j = 1, 2$, such that the set $h_1 \cap h_2$ has finitely manus points, and $\iota([l_1], [l_2])$ is defined as the algebraic intersection number $\iota(h_1, h_2)$ between these smooth arcs that is $\iota([l_1], [l_2]) = \iota(h_1, h_2)$.

Similarly let $A \subset S$ be a topological annulus. Let l be an oriented Jordan arc that has one endpoint in each $\partial_0(A)$ and $\partial_1(A)$ (in particular this implies that the endpoints of l are accessible points in the boundary of A). The homotopy class $[l]$ (modulo the endpoints) of l is the collection of all such arcs homotopic to l in A, that have the same endpoints as l, and with the corresponding orientations. For two such arcs l_1 and l_2, by $\iota([l_1], [l_2]) \in \mathbf{Z}$ we denote the algebraic intersection number between the two homotopy classes. This is well defined for the same reasons as above. Endow M with a complex structure and let $\Phi : N \to A$ be a surjective

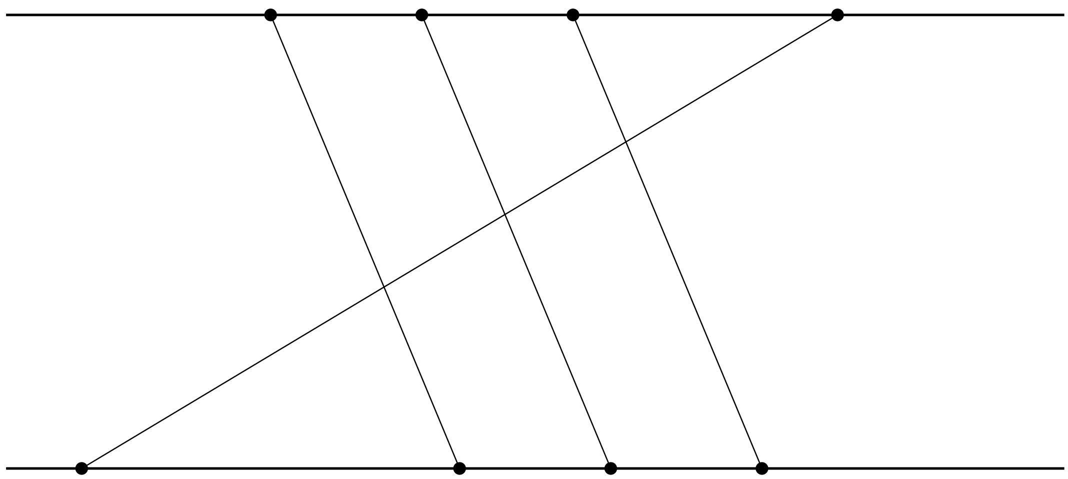

FIGURE 5. In this case the arcs h_1 and h_2 are straight lines in P.

conformal map. Then $\Phi^{-1}(l)$ is a Jordan arc that connects the two boundary circles of N. We have $\iota([l_1],[l_2]) = \iota([\Phi^{-1}(l_1)],[\Phi^{-1}(l_2)])$.

The following proposition is elementary. The proof is left to the reader (see Figure 5).

Proposition 3.3. *Let $A \subset N$ be a topological annulus homotopic to N (we allow that $A = N$). Let $l_1, l_2 \subset A$ be two oriented Jordan arcs, each of them having one of its endpoints in each $\partial_0(A)$ and $\partial_1(A)$. Let $z_1, w_1 \in \overline{P}$ be the endpoints of a lift of l_1 to P and let $z_2, w_2 \in \overline{P}$ be the endpoints of a lift of l_2 to P. Let $h_1 \subset P$ be any smooth oriented arc with the endpoints z_1, w_1 and let $h_2 \subset P$ be any smooth oriented arc with the endpoints z_2, w_2. Then*

$$\iota([l_1],[l_2]) = \sum_{k \in \mathbf{Z}} \iota(T^k(h_1), h_2).$$

Proposition 3.4. *Let $\widetilde{f} : \overline{N} \to \overline{N}$ be a homeomorphism that fixes setwise the boundary circles of N. Let l be any oriented Jordan arc in N connecting two boundary circles of N. Then*

$$\left| \iota([l],[\widetilde{f}^n(l)]) - n\rho(\widetilde{f}, N) \right| < 8, \tag{4}$$

for every $n \in \mathbf{N}$. In particular

$$\rho(\widetilde{f}, N) = \lim_{n \to \infty} \frac{\iota([l],[\widetilde{f}^n(l)])}{n}.$$

Remark. The same proposition holds for a homeomorphism $\widetilde{f} : \overline{A} \to \overline{A}$, where $A \subset S$ is a topological annulus.

Proof. Let $\widehat{f}$ be a lift of $\widetilde{f}$ to P. Let $\widehat{l}$ be a single lift of l to P. Let $z_0 \in \partial_0(P)$ and $z_1 \in \partial_1(P)$, be the endpoints of $\widehat{l}$. We compute the algebraic intersection number between the classes $[l]$ and $[\widetilde{f}^n(l)]$ as follows. Replace the arcs $\widehat{l}$ and $\widehat{f}^n(\widehat{l})$ by the straight lines that have the same endpoints as these two arcs, and denote these straight arcs by h and h_n respectiveley. Then $\iota([l],[\widetilde{f}^n(l)])$ is equal to the signed number of different translates of h that intersect h_n in P (see the previous proposition and Figure 5). The sign is equal to the intersection number between a single translate of h (that intersects h_n) and h_n. Then for every $n \in \mathbf{N}$ we have

$$\Big| \big(Re(\widehat{f}^n(z_1)) - Re(\widehat{f}^n(z_0))\big) - (Re(z_1) - Re(z_0)) - \iota([l],[\widetilde{f}^n(l)]) \Big| \leq 2.$$

Combining this inequality with Proposition 3.2 we obtain (4). Divide this inequality by n, and let $n \to \infty$. This proves the rest of the proposition. □

3.3. **Long range Lipschitz maps on the strip.** We have the following definition.

Definition 3.4. *Let $(X, \mathbf{d})$ be a metric space and let $x_1, x_2 \in X$. Let F be a group of homeomorphisms of X. We say that the group F is K long range Lipschitz on the pair of points x_1, x_2, if $\mathbf{d}(f(x_1), f(x_2)) \leq K$ for every homeomorphism $f \in F$.*

Remark. The constant K in the above definition may depend on the choice of $x_1, x_2 \in X$.

Lemma 3.1. *Let $\widetilde{f} : \overline{N} \to \overline{N}$ be a homeomorphism that setwise preserves the sets $\partial_0(N)$ and $\partial_1(N)$, and such that $\widetilde{f} \circ \widetilde{e} = \widetilde{e} \circ \widetilde{f}$. Also, let $C \subset N$ be a relatively closed set such that $\widetilde{f}(C) = \widetilde{e}(C) = C$, and every connected component of C is compactly contained in N. Let $\widehat{f} : \overline{P} \to \overline{P}$ be a lift of $\widetilde{f}$ and assume that for every $z_1, z_2 \in (P \setminus \widehat{C})$ the cyclic group generated by $\widehat{f}$ is K long range Lipschitz on the pair z_1, z_2, for some constant $K = K(z_1, z_2) > 0$ that depends on z_1, z_2 (here $\widehat{C}$ is the lift of C to P). Then $\rho(\widetilde{f}, N) = 0$.*

Proof. There exists an integer $m \in \mathbf{Z}$, so that $\widehat{f} \circ \widehat{e} = T^m \circ \widehat{e} \circ \widehat{f}$. Note that $\widehat{f}(\widehat{C}) = \widehat{C} = \widehat{e}(\widehat{C})$. Since $\widehat{f}$ commutes with the translation $T(z)$ we conclude that each $\widehat{f}^n$, $n \in \mathbf{Z}$, is uniformly continuous on $\overline{P}$. That is, for a fixed $n \in \mathbf{Z}$ and for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n) > 0$ so that for every two points $z_1, z_2 \in \overline{P}$ such that $|z_1 - z_2| \leq \delta$, we have $|\widehat{f}^n(z_1) - \widehat{f}^n(z_2)| \leq \epsilon$.

Let
$$H(r) = \bigcup_{k \in \mathbf{Z}} \widehat{f}^k(P(r) \setminus \widehat{C}).$$
Since $\widehat{f}$ commutes with $T(z)$ we have $T(H(r)) = H(r)$, where $T(z)$ is the translation.

Proposition 3.5. *With the notation stated above we have the following. Assume that for some $1 < r_1 < r_0$ we have that $H(r_1)$ has an accumulation point on ∂P. Then $\rho(\widetilde{f}, N) = 0$.*

Proof. Since $\widehat{f}$ commutes with $T(z)$ there exists $w_1 \in (P(r_1) \setminus \widehat{C})$ such that $0 \leq Re(w_1) < 1$, and such that $\widehat{f}^{n_k}(w_1)$ converges to ∂P, where n_k is a sequence of integers. Without loss of generality we may assume that

$$\lim_{n_k \to \infty} Im(\widehat{f}^{n_k}(w_1)) = \frac{\log r_0}{2\pi},$$

that is the sequence $\widehat{f}^{n_k}(w_1)$ converges to $\partial_1(P)$. Let $w_0 = \widehat{e}(w_1)$ (note that w_0 does not belong to $\widehat{C}$). Then $\widehat{f}^{n_k}(w_0)$ converges to $\partial_0(P)$ because $\widehat{e}(\partial_1(P)) = \partial_0(P)$.

Since $w_1, w_0 \in (P \setminus \widehat{C})$, we have that the group generated by $\widehat{f}$ is K long range Lipschitz on the pair w_0, w_1, for some constant $K > 0$. This implies that for every $k \in \mathbf{Z}$ the Euclidean distance between the points $\widehat{f}^k(w_1)$ and $\widehat{f}^k(w_0)$ is bounded above by the constant K, that is

$$|\widehat{f}^k(w_1) - \widehat{f}^k(w_0)| \leq K,\, k \in \mathbf{Z}. \tag{5}$$

Assume now that $\rho(\widetilde{f}, N) \neq 0$. Let $m_0 \in \mathbf{Z}$ be such that

$$m_0 \rho(\widetilde{f}, N) > 2K + 11. \tag{6}$$

Since $\widehat{f}^{m_0}$ is uniformly continuous on $\overline{P}$, there exists $\delta' > 0$ such that for every two points $z, z' \in \overline{P}$, with $|z - z'| \leq \delta'$, we have

$$|\widehat{f}^{m_0}(z) - \widehat{f}^{m_0}(z')| \leq \frac{1}{2}. \tag{7}$$

Let $\delta = \min\{\frac{1}{3}, \delta'\}$.

Let n_k be large enough so that the point $\widehat{f}^{n_k}(w_1)$ is within the Euclidean distance δ from $\partial_1(P)$. Then $\widehat{f}^{n_k}(w_0)$ is within the Euclidean distance δ from $\partial_0(P)$ (since $\widehat{e}$ is an isometry). Let $z_1 \in \partial_1(P)$ be a point such that $|\widehat{f}^{n_k}(w_1) - z_1| \leq \delta$, and $z_0 \in \partial_0(P)$ be a point such that $|\widehat{f}^{n_k}(w_0) - z_0| \leq \delta$. We have

$$|Re(z_1) - Re(z_0)| \leq |Re(\widehat{f}^{n_k}(w_1)) - Re(\widehat{f}^{n_k}(w_0))| + |\widehat{f}^{n_k}(w_1) - z_1| + |\widehat{f}^{n_k}(w_0) - z_0| \leq$$
$$\leq K + 2\delta < K + 2.$$

Then it follows from Proposition 3.2 that

$$\left| Re(\widehat{f}^{m_0}(z_1)) - Re(\widehat{f}^{m_0}(z_0)) - m_0\rho(\widehat{f}, P) \right| \leq (K+2) + 6 < K + 9,$$

which together with (6) implies that

$$|Re(\widehat{f}^{m_0}(z_1)) - Re(\widehat{f}^{m_0}(z_0))| \geq m_0\rho(\widehat{f}, P) - K - 9 > K + 2.$$

Again from the triangle inequality and from (7) we get

$$|Re(\widehat{f}^{(n_k+m_0)}(w_1)) - Re(\widehat{f}^{n_k+m_0}(w_0))| \geq |Re(\widehat{f}^{m_0}(z_1)) - Re(\widehat{f}^{m_0}(z_0))| -$$
$$-|Re(\widehat{f}^{(n_k+m_0)}(w_1)) - \widehat{f}^{m_0}(z_1)| - |Re(\widehat{f}^{(n_k+m_0)}(w_0)) - \widehat{f}^{m_0}(z_0)| \geq K + 2 - 2\frac{1}{2} = K + 1.$$

But this contradicts (5). $\square$

It remains to consider the case when every H_r is a subset of $P(r')$ for some $r' < r_0$ (here r' depends on H_r). The proof is by contradiction. From now on we assume that $\rho(\widehat{f}, P) \neq 0$. Let $m \in \mathbf{Z}$ so that $\rho(\widehat{f}^m, P) > 10$. Then for any pair of points $z_1, z_2 \in (P \setminus \widehat{C})$ the group generated by $\widehat{f}^m$ is also K long range Lipschitz on the pair z_1, z_2. So we may assume that $\rho(\widehat{f}, P) > 10$.

Since $\widehat{f}$ is uniformly continuous on $\overline{P}$, there exists $\delta' > 0$ such that for every two points $z, z' \in \overline{P}(r_0)$, with $|z - z'| \leq \delta'$, we have $|\widehat{f}(z) - \widehat{f}(z')| \leq \frac{1}{2}$. Let $\delta = \min\{\frac{1}{3}, \delta'\}$. Let $r_1 < r_0$ be close enough to r_0 so that for every $z \in P \setminus P(r_1)$, we have that the distance between z and ∂P is less than δ. Then the same is true for every point in $P \setminus H_{r_1}$ since $P(r_1) \subset H_{r_1}$.

It follows from our assumption on the set C that no connected component of $\widehat{C}$ in P accumulates on the boundary of P. Therefore we can find $w_1 \in P \setminus (H_{r_1} \cup \widehat{C})$ such that $Re(w_1) = 0$. Let $\widehat{e}(w_1) = w_0$. Then $w_0 \in P \setminus H_{r_1}$ and $Re(w_0) = 0$. Let

$z_1 = \frac{\log r_0}{2\pi}$ and $z_0 = -\frac{\log r_0}{2\pi}$. Then the Euclidean distance between z_i and w_i is less than δ, for $i = 0, 1$.

We now show that $|\widehat{f}^n(z_i) - \widehat{f}^n(w_i)| < 2n$. The statement is true for $n = 1$ by the choice of δ. We assume that it is true for n and prove it for $n+1$. Let $\zeta_i^n \in \partial_i(P)$ be the point with the same x-coordinate as $\widehat{f}^n(w_i)$. Then $|\zeta_i^n - \widehat{f}^n(w_i)| < \delta$ since $\widehat{f}^n(w_i) \in P \setminus H_{r_1}$. This implies that

$$|\widehat{f}(\zeta_i^n) - \widehat{f}^{n+1}(w_i)| < \frac{1}{2}. \tag{8}$$

Since $|\widehat{f}^n(z_i) - \widehat{f}^n(w_i)| < 2n$ we conclude that

$$|\zeta_i^n - \widehat{f}^n(z_i)| = |Re\left(\widehat{f}^n(w_i) - \widehat{f}^n(z_i)\right)| < 2n.$$

This implies that $|\widehat{f}(\zeta_i^n) - \widehat{f}^{n+1}(z_i)| < 2n + 1$. The triangle inequality and (8) yield

$$|\widehat{f}^{n+1}(z_i) - \widehat{f}^{n+1}(w_i)| < |\widehat{f}^{n+1}(z_i) - \widehat{f}(\zeta_i^n)| + |\widehat{f}(\zeta_i^n) - \widehat{f}^{n+1}(w_i)| \leq 2n + \frac{3}{2} < 2(n+1).$$

This proves the induction statement.

We have that $|\widehat{f}^n(w_1) - \widehat{f}^n(w_0)| \leq K$, for every $n \in \mathbf{Z}$, where K is such that the group generated by $\widehat{f}$ is K long range Lipschitz on w_0 and w_1. By the above induction statement and from the triangle inequality we have $|\widehat{f}^n(z_1) - \widehat{f}^n(z_0)| \leq K + 4n$. Since $\rho(\widehat{f}, P) > 10$ we obtain a contradiction from Proposition 3.2.

$\square$

3.4. **Fixed points of a strip homeomorphism.** Let $\Omega \subset P$ be a simply connected domain that is invariant under the translation $T(z) = z + 1$ (this means that Ω is the lift of a topological annulus $A \subset N$ with respect to the covering map $P \to N$). Then there exists $1 < r_1 \leq r_0$ and a conformal map $\Phi : P(r_1) \to P$, such that Φ commutes with the translation $T(z) = z + 1$ and $\Phi(P(r_1)) = \Omega$.

Definition 3.5. *Let $A \subset N$ be a topological annulus that is homotopic to N. We say that A is a faithful domain if*

- *We have* $\text{int}(\overline{A}) = A$*, that is the interior of the closure of A coincides with A.*
- *Let E be a connected component of the set $N \setminus \overline{A}$. Then the frontier ∂E of E is not contained in $\overline{A}$.*

If $\Omega \subset P$ is a a simply connected domain that is invariant under the translation $T(z)$ we say that Ω is a faithful domain if the corresponding annulus $A \subset N$ is a faithful domain. Equivalently this means that the interior of the closure of Ω coincides with Ω, and that if E is a connected component of the set $P \setminus \overline{\Omega}$ then the relative frontier ∂E of E is not contained in $\overline{\Omega}$ (by the term relative frontier of E we mean the frontier points of E except ∞). In Figure 6 we have an example of a domain $\Omega \subset P$ that is a simply connected domain, invariant under the translation $T(z)$, but that is not faithful.

Remark. Let $A \subset N$ be a topological annulus homotopic to N. Let Ω be the lift of A to P. Then A is faithful if and only if Ω is faithful.

Proposition 3.6. *Suppose that $\Omega \subset P$ is a faithful and simply connected set that is invariant under the translation $T(z)$. Let $\Phi : P(r_1) \to P$, be a conformal map such that Φ commutes with $T(z)$ and $\Phi(P(r_1)) = \Omega$. Then there exists a constant $C > 0$ such that $|\Phi(z) - z| \leq C$, for every $z \in P(r_1)$.*

Remark. This proposition does not hold if Ω is not faithful (see Figure 6).

Proof. Let $\mathcal{D} \subset P(r_1)$ be a rectangle fundamental domain for the action of $T(z)$ on $P(r_1)$. If we prove that $\Phi(\mathcal{D})$ has a finite Euclidean diameter in P then the proposition follows since Φ commutes with the translation $T(z)$.

The proof is by contradiction. Assume that $\Phi(\mathcal{D})$ has an infinite diameter. Then there exists a sequence $z_n \in \mathcal{D}$ such that the Euclidean distance between $\Phi(z_1)$ and $\Phi(z_n)$ goes to ∞ as $n \to \infty$. Set $w_n = \Phi(z_n)$. This implies that $Re(w_n) \to \infty$. Without loss of generality we may assume that $Re(w_n) \to +\infty$.

Let l_∞ be the hyperbolic geodesic ray in $P(r_1)$ that starts at z_1 and ends at $+\infty$. Let l_n be the hyperbolic geodesic ray that starts at z_1, and which contains z_n. After passing onto a subsequence if necessary, we have that z_n converges to some point in $\overline{\mathcal{D}}$. Let l_* be the limit of the geodesic rays l_n. The geodesic ray l_* starts at z_1 and by $z_* \in \partial P(r_1)$ we denote the endpoint of l_* on $\partial P(r_1)$ (without loss of generality we may assume that $z_* \in \partial_1(P(r_1))$). Note that $l_* \neq l_\infty$. Then $l_\infty \cup l_*$ divides $P(r_1)$ into two simply connected sets D_1 and D_2. We have that z_* divides $\partial_1(P(r_1))$ into two Euclidean rays γ_1 and γ_2. The set D_1 contains γ_1 in its boundary and D_2 contains γ_2 in its boundary. Moreover the boundary of D_1 is $l_\infty \cup l_* \cup \gamma_1 \cup \{+\infty\}$.

Let $l'_\infty = \Phi(l_\infty)$ and $l'_* = \Phi(l_*)$. Since Φ commutes with the translation we see that $\Phi(+\infty) = +\infty$. We conclude that l'_∞ has $+\infty$ as its endpoint. It follows from the assumption $Re(w_n) \to +\infty$ that l'_* has $+\infty$ as its endpoint as well. However the curves l'_∞ and l'_* do not coincide because the curves l_∞ and l_* do not coincide either. Then the set $P \setminus (l'_\infty \cup l'_*)$ has two simply connected components E_1 and E_2, and

$$\partial E_1 = l'_\infty \cup l'_*.$$

Moreover $\Phi(D_1) \subset E_1$ and $\Phi(D_2) \subset E_2$.

Let $\zeta_1 \in \gamma_1$, be an accessible point for Φ, and let α_1 be the hyperbolic geodesic ray from z_1 to ζ_1. Then $\Phi(\alpha_1)$ is a finite diameter arc in Ω. Moreover $\Phi(\zeta_1) \in E_1$ because $\Phi(D_1) \subset E_1$ and $\Phi(\zeta_1)$ does not belong to $l'_\infty \cup l'_*$. This shows that E_1 is not a subset of Ω. If the set $E_1 \setminus \overline{\Omega}$ is non-empty then every connected component O of this set is also a connected component of the set $P \setminus \overline{\Omega}$. Moreover ∂O is contained in $\overline{\Omega}$ which is impossible since Ω is faithful. Therefore we have that the set $E_1 \setminus \overline{\Omega}$ is empty, that is $E_1 \subset \overline{\Omega}$. But since Ω is faithful and since E_1 is an open set we conclude that $E_1 \subset \Omega$. This is a contradiction since $\Phi(\zeta_1) \in E_1$ and since $\Phi(\zeta_1)$ does not belong to Ω. So the set $\Phi(\mathcal{D})$ has a finite Euclidean diameter.

□

Fix a homeomorphism $\widehat{f} : \overline{P} \to \overline{P}$ that commutes with the translation $T(z)$. Let $1 < r_1 < r_0$ and let $\Phi : P(r_1) \to P = P(r_0)$ be a conformal map (it is conformal onto its image that is we do not assume that $\Phi(P(r_1)) = P$) that commutes with the translation $T(z)$, that is $\Phi(z+1) = \Phi(z) + 1$. Set $\Omega = \Phi(P(r_1))$ and assume that $\widehat{f}$ setwise preserves Ω. Since $\widehat{f}$ is a homeomorphism of $\overline{\Omega}$, we conclude that the map $\widehat{g} = \Phi^{-1} \circ \widehat{f} \circ \Phi$ is a homeomorphism of $\overline{P(r_1)}$. We define $\rho(\widehat{f}, \Omega) = \rho(\widehat{g}, P(r_1))$.

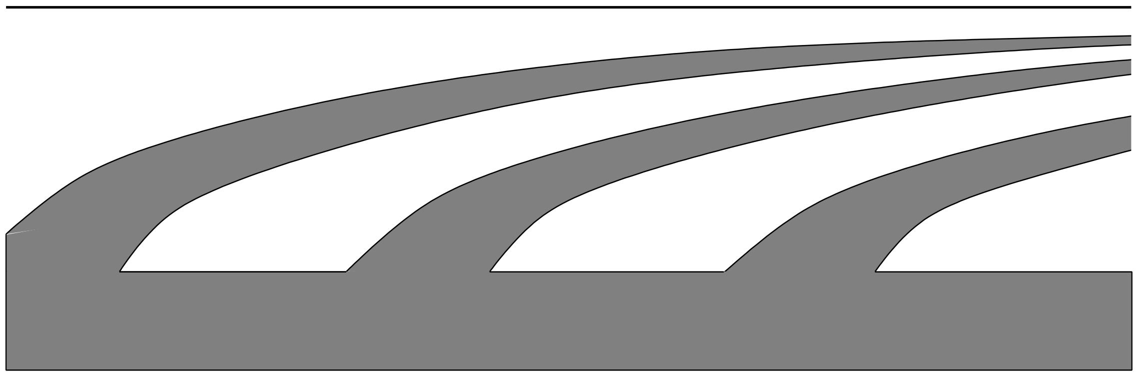

FIGURE 6. This is an example of a simply connected domain that is invariant under the translation $T(z)$ but that is not faithful.

First we prove that under certain conditions on Ω and $\widehat{f}$, the map $\widehat{g}$ has a fixed point on $\partial P(r_1)$.

Proposition 3.7. *Assuming the above notation we have the following. Suppose that Ω is faithful. If there exists a compact set $K \subset \partial\Omega$ which is setwised fixed by $\widehat{f}$, then the homeomorphism $\widehat{g}$ has a fixed point on $\partial P(r_1)$.*

Remark. It seems that this statement holds even if Ω is not faithful, but in that case the proof would be more involved.

Proof. Let $\mathcal{D} \subset P(r_1)$ be a rectangle fundamental domain for the action of $T(z)$ on $P(r_1)$. In the previous proposition we have proved that $\Phi(\mathcal{D})$ has a finite Euclidean diameter in P. We claim that the set

$$\bigcup_{k\in\mathbb{Z}} T^k(\overline{\Phi(\mathcal{D})}),$$

covers $\overline{\Omega}$. Let $w_0 \in \overline{\Omega}$ and let $w_n \in \Omega$ be a sequence that converges to w_0. Then w_n is a bounded sequence and since $\Phi(\mathcal{D})$ has a finite diameter we conclude that finitely many translates of $\Phi(\mathcal{D})$ contain all points w_n. This implies that w_0 belongs to the closure of some translate of $\overline{\Phi(\mathcal{D})}$.

Since K is a compact set, only finitely many translates of $\overline{\Phi(\mathcal{D})}$ intersect K. Let $M_K = \{m \in \mathbf{Z} : T^m\big(\overline{\Phi(\mathcal{D})}\big) \cap K \neq \emptyset\}$. Then the homeomorphism $\widehat{f}$ setwise fixes the set $\bigcup_{m\in M_K} T^m\big(\overline{\Phi(\mathcal{D})}\big)$. This implies that the homeomorphism $\widehat{g}$ setwise fixes

$$\partial_1(P(r_1)) \bigcap \left(\bigcup_{m\in M_K} T^m(\overline{\mathcal{D}}) \right).$$

Let $z_1, z_2 \in \partial_1(P(r_1))$ so that $Re(z_1)$ and $Re(z_2)$ are respectively the infimum and the supremum of the set

$$\partial_1(P(r_1)) \bigcap \left(\bigcup_{m\in M_K} T^m(\overline{\mathcal{D}}) \right).$$

Then $-\infty < Re(z_1) \le Re(z_2) < +\infty$. We have that the homeomorphism $\widehat{g}$ fixes z_1 and z_2. Therefore $\widehat{g}$ has a fixed point. $\square$

The following proposition is the version of Proposition 3.2 for faithful domains.

Proposition 3.8. *Let $1 < r_1 \le r_0$ and let $\Phi : P(r_1) \to P$ be a conformal map that commutes with the translation $T(z)$. Set $\Phi(P(r_1)) = \Omega$ and suppose that Ω is faithful domain. Let $\widehat{f} : \overline{P} \to \overline{P}$ be a homemorphism that setwise fixes the boundary lines of P, that commutes with the translation $T(z)$, and such that $\widehat{f}(\Omega) = \Omega$. Then there exists a constant $K > 0$, such that for every $w_0 \in \partial_0(\Omega)$, $w_1 \in \partial_1(\Omega)$, we have*

$$\Big|\big(Re(\widehat{f}^n(w_1)) - Re(\widehat{f}^n(w_0))\big) - (Re(w_1) - Re(w_0)) - n\rho(\widehat{f}, \Omega)\Big| < K.$$

for every $n \in \mathbf{N}$.

Proof. Recall that $\widehat{g} = \Phi^{-1} \circ \widehat{f} \circ \Phi$. Let C be the constant from Proposition 3.6. We show that the proposition holds for $K = 4C + 6$.

Assume first that w_0 and w_1 are accessible points. Let $\Phi^{-1}(w_i) = z_i \in \partial_i(P(r_1))$. Then by Proposition 3.2 we have

$$\big|\big(Re(\widehat{g}^n(z_1)) - Re(\widehat{g}^n(z_0))\big) - (Re(z_1) - Re(z_0)) - n\rho(\widehat{g}, P(r_1))\big| < 6.$$

Since

$$|w_i - z_i|,\ |\widehat{f}^n(w_i)) - \widehat{g}^n(z_i))| \le C, \quad i = 0, 1$$

we conclude that

$$\Big|\big(Re(\widehat{f}^n(w_1)) - Re(\widehat{f}^n(w_0))\big) - (Re(w_1) - Re(w_0)) - n\rho(\widehat{f}, \Omega)\Big| < 4C + 6 = K.$$

The proposition now follows from the fact that the accessible points are dense in $\partial\Omega$.

$\square$

We have

Lemma 3.2. *Let $\widetilde{f} : \overline{N} \to \overline{N}$ be a homeomorphism that setwise preserves the boundary circles of N. Let $A_i \subset N$, $i = 0, 1$, be two mutually disjoint toplogical annuli that are homotopic to N such that $\partial_0(A_0) = \partial_0(N)$ and $\partial_1(A_1) = \partial_1(N)$. Suppose that the domains A_i, $i = 0, 1$ are faithful domains. Let $Q = N \setminus (A_0 \cup A_1)$ and suppose $\widetilde{f}(A_i) = A_i$. If $\operatorname{int}(Q)$ (the interior of Q) does not contain a component that is a topological annulus homotopic to N then*

$$\rho(\widetilde{f}, A_0) + \rho(\widetilde{f}, A_1) = \rho(\widetilde{f}, N).$$

If A is a component of $\operatorname{int}(Q)$ that is a topological annulus homotopic to N, and if A is a faithful domain, then

$$\rho(\widetilde{f}, A_0) + \rho(\widetilde{f}, A_1) + \rho(\widetilde{f}, A) = \rho(\widetilde{f}, N).$$

Proof. By $\widehat{f} : \overline{P} \to \overline{P}$ we denote a lift of $\widetilde{f}$. Assume first that $\operatorname{int}(Q)$ does not contain a component that is a toplogical annulus homotopic to N. Since $\operatorname{int}(Q)$ does not contain a component that is a toplogical annulus homotopic to N we conclude that there exists a point $r \in Q$ such that $r \in \partial A_0 \cap \partial A_1$. This point r is fixed from now on. Let $s_i \in \partial_i(N)$ be any two points. Let $\widehat{s}_i$, $i = 0, 1$, and $\widehat{r}$ be lifts to $\overline{P}$ of the points s_i, $i = 0, 1$, and r, respectively. Let Ω_i, $i = 0, 1$, be the lifts of A_i to P. Then $\widehat{r} \in \partial\Omega_0 \cap \partial\Omega_1$. We apply Proposition 3.8 to the pair $\widehat{s}_0$ and $\widehat{r}$, and then to the pair $\widehat{s}_1$ and $\widehat{r}$. This proves the proposition in this case.

The case when Q contains a component that is a toplogical annulus homotopic to N is handled in a similar way. Assume that $\mathrm{int}(Q)$ has a connected component A that is a topological annulus homotopic to N. Since Q is connected such A is unique. Moreover there exist points $r_i \in \partial A_i \cap \partial A$, $i = 0, 1$. Let $s_i \in \partial_i(N)$ be any two the points. Since the lifts of all three annuli are faithful domains the proof follows in the same way as above.

□

3.5. **The K-idle set of an annulus homeomorphism.** We have the following definition.

Definition 3.6. *Let $\widetilde{f} : \overline{N} \to \overline{N}$ be a homeomorphism that setwise preserves the sets $\partial_0(N)$ and $\partial_1(N)$. Let $w \in N$ and let $K > 0$. We say that the point w is K-idle for $\widetilde{f}$ if there exists a lift $\widehat{f} : \overline{P} \to \overline{P}$ of $\widetilde{f}$ such that*

$$|Re(\widehat{f}^n(z)) - Re(\widehat{f}^m(z))| \leq K, \tag{9}$$

for every $m, n \in \mathbf{Z}$, and for every $z \in P$ that is a lift of w. The set of all K-idle points for $\widetilde{f}$ is denoted by $\mathcal{I}(K, \widetilde{f}, N)$.

Since $\widehat{f}$ commutes with the translation $T(z)$ we have that the condition (9) holds for one lift $z \in P$ of w if and only if it holds for every such lift $z \in P$ of w. If $w \in \mathcal{I}(K, \widetilde{f}, N)$ then there exists a unique lift $\widehat{f} : \overline{P} \to \overline{P}$ such that (9) holds. Also it follows from the definition that $\widetilde{f}(\mathcal{I}(K, \widetilde{f}, N)) = \mathcal{I}(K, \widetilde{f}, N)$ (this follows from (9)).

Remark. Let $\widetilde{e}$ be a conformal involution of N and assume that $\widetilde{f}$ commutes with $\widetilde{e}$. Since every lift $\widehat{e} : P \to P$ of $\widetilde{e}$ is an Euclidean isometry, we conclude that $\widetilde{e}(\mathcal{I}(K, \widetilde{f}, N)) = \mathcal{I}(K, \widetilde{f}, N)$.

Let $w_i \in \mathcal{I}(K, \widetilde{f}, N)$, $i = 1, 2$, and let $\widehat{f}_i$ denote the corresponding lift so that the condition (9) holds for every lift of w_i to P. We say that w_1 and w_2 are equivalent, $w_1 \sim w_2$, if $\widehat{f}_1 = \widehat{f}_2$.

Proposition 3.9. *The set $\mathcal{I}(K, \widetilde{f}, N)$ is a relatively closed subset of N. Let Q be a connected component of $\mathcal{I}(K, \widetilde{f}, N)$. Then every two points in Q are equivalent. If in addition we have $\rho(\widetilde{f}, N) \neq 0$ then $\overline{Q}$ does not connect the two boundary circles $\partial_0(N)$ and $\partial_1(N)$.*

Proof. Let $\widehat{f} : \overline{P} \to \overline{P}$ be a lift of $\widetilde{f}$ and fix $n \in \mathbf{Z}$. Since $\widehat{f}$ commutes with the translation $T(z)$ we conclude that $\widehat{f}^n$ is uniformly continuous on $\overline{P}$. That is for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, n) > 0$ so that for every two points $z_1, z_2 \in \overline{P}$ such that $|z_1 - z_2| \leq \delta$, we have $|\widehat{f}^n(z_1) - \widehat{f}^n(z_2)| \leq \epsilon$. Since every two lifts of $\widetilde{f}$ differ by a translation (which is an isometry in the Euclidean metric) we see that these constants do not depend on the choice of the lift $\widehat{f}$.

Fix $n_0 \in \mathbf{N}$ such that $n_0 > K + 3$ and let $\epsilon_0 = 1$. Set $\delta_0 = \min\{\delta(\epsilon_0, n_0), 1\}$. Then for every lift $\widehat{f} : \overline{P} \to \overline{P}$ we have

$$|\widehat{f}^{n_0}(z_1) - \widehat{f}^{n_0}(z_2)| \leq 1, \qquad \text{for every} \quad |z_1 - z_2| \leq \delta_0. \tag{10}$$

Let $w_i \in \mathcal{I}(K, \widetilde{f}, N)$. Let $\widehat{f}_i : \overline{P} \to \overline{P}$ be the lift of $\widetilde{f}$ such that the condition (9) holds with respect to $\widehat{f}_i$. Then $w_1 \sim w_2$ if and only if $\widehat{f}_1 = \widehat{f}_2$. Assume that we can choose lifts $z_1, z_2 \in P$, of w_1 and w_2 respectively, such that $|z_1 - z_2| \leq \delta_0$. Then we show that $\widehat{f}_1 = \widehat{f}_2$. This implies that w_1 and w_2 are equivalent. In turn this shows that the set $\mathcal{I}(K, \widetilde{f}, N)$ is a relatively closed subset of N and that every two points in Q are equivalent.

Next we show that $\widehat{f}_1 = \widehat{f}_2$ providing that $|z_1 - z_2| \leq \delta_0$. Assume that $\widehat{f}_1 \neq \widehat{f}_2$. Then $\widehat{f}_2 = T^k \circ \widehat{f}_1$ for some $k \in \mathbf{Z}$, and $k \neq 0$. Since the condition (9) holds for z_2 and $\widehat{f}_2$ we have

$$|Re(\widehat{f}_2^{n_0}(z_2)) - Re(z_2)| \leq K.$$

On the other hand we have

$$|Re(\widehat{f}_2^{n_0}(z_2)) - Re(z_2)| = |kn_0 + (Re(\widehat{f}_1^{n_0}(z_2)) - Re(z_2))| \geq |kn_0| - |Re(\widehat{f}_1^{n_0}(z_2)) - Re(z_2)|.$$

It follows from (10) and the triangle inequality that (note that $\delta_0 \leq 1$)

$$|Re(\widehat{f}_1^{n_0}(z_2)) - Re(z_2)| \leq$$

$$\leq |Re(\widehat{f}_1^{n_0}(z_2)) - Re(\widehat{f}_1^{n_0}(z_1))| + |Re(\widehat{f}_1^{n_0}(z_1)) - Re(z_1)| + |Re(z_1) - Re(z_2)| \leq K + 2,$$

that is

$$|Re(\widehat{f}_2^{n_0}(z_2)) - Re(z_2)| \geq |kn_0| - (K+2) > |k|(K+3) - (K+2) \geq K+1.$$

But this is a contradiction so we conclude that $\widehat{f}_1 = \widehat{f}_2$.

Assume that $\rho(\widetilde{f}, N) \neq 0$. Let Q be a connected component of $\mathcal{I}(K, \widetilde{f}, N)$ and let $Q_1 \subset P$ be a connected component of the lift of Q to P. Then by the previous argument we have that (9) holds for every $z \in Q_1$. By the continuity we have that (9) holds for every $z \in \overline{Q_1}$. If Q connects the two boundary circles of N then Q_1 connects the two boundary lines of P. Let $z_i \in \partial_i(P)$, $i = 0, 1$, be such that $z_i \in \overline{Q}_1$. Then for every $n \in \mathbf{Z}$ we have

$$|Re(\widehat{f}^n(z_1)) - Re(z_1)| \leq K, \quad i = 0, 1.$$

Since $\rho(\widetilde{f}, N) \neq 0$ this contradicts Proposition 3.2.

□

As above let S denote a compact Riemann surface (either closed or with boundary). Let $A \subset S$ be a topological annulus. Let $\widetilde{f} : \overline{S} \to \overline{S}$ be a homeomorphism such that $\widetilde{f}(A) = A$. Let $\Phi : N \to A$ be a conformal map and set $\widetilde{g} = \Phi^{-1} \circ \widetilde{f} \circ \Phi$. Since $\widetilde{f}$ is a homeomorphism of $\overline{A}$ it follows that $\widetilde{g}$ is a homeomorphism of $\overline{N}$. We define the corresponding set $\mathcal{I}(K, \widetilde{f}, A) \subset A$ to be equal to $\Phi(\mathcal{I}(K, \widetilde{g}, N))$.

Proposition 3.10. *Let $\widetilde{f} : \overline{N} \to \overline{N}$ be a homeomorphism that has fixed points on both boundary circles of N and that is homotopic to the standard twist modulo its fixed points on ∂N. Let $\widetilde{e}$ be a conformal involution of N that exchanges the boundary circles of N and suppose that $\widetilde{f}$ commutes with $\widetilde{e}$. Let $A \subset N$ be a topological annulus homotopic to N, such that $\widetilde{e}(A) = \widetilde{f}(A) = A$, and assume that $\rho(\widetilde{f}, A)$ is an odd integer. Fix $K > 0$ and suppose that there exists a connected component Q of the set set $\mathcal{I}(K, \widetilde{f}, A)$ such that Q is compactly contained in A and that Q separates the two frontier components of A. Then there exists a topological annulus $A_1 \subset A$, and $A \neq A_1$, such that $\widetilde{e}(A_1) = \widetilde{f}(A_1) = A_1$, and $\rho(\widetilde{f}, A_1)$ is an odd integer.*

Proof. The set Q is compact in A and it separates the two frontier components of A (and therefore Q separates the two boundary circles of N). Set $Q_1 = \widetilde{e}(Q)$. Let B' be union of $Q \cup Q_1$ and every connected component of the set $N \setminus (Q \cup Q_1)$ whose boundary is contained in $Q \cup Q_1$. Then B' is a closed set that is compactly contained in A. Then the set $N \setminus B'$ is a disjoint union of two topological annuli A'_0 and A'_1 that satisfy the conditions

- $\partial_0(A'_0) = \partial_0(N)$ and $\partial_1(A'_1) = \partial_1(N)$,
- $\partial_1(A'_0), \partial_0(A'_1) \subset (Q \cup Q_1)$.

We construct the sets A_i, $i = 0, 1$, as follows. Let A''_i be the union of A'_i and every connected component of the set $N \setminus \overline{A'}_i$ whose boundary is contained in $\overline{A'}_i$. Set $A_i = \operatorname{int}(A''_i)$. Then $A_1 = \widetilde{e}(A_0)$. Note that $A_0 \cap A_1 = \emptyset$ and each A_i is a topological annulus that is a faithful set in the sense of Definition 3.5. Set $B = N \setminus (A_0 \cup A_1)$. Note that $\partial_1(A_0)$, $\partial_0(A_1) \subset (Q \cup Q_1)$.

Recall that we are assuming that $\widetilde{f}$ has a fixed point on both boundary circles of N. This implies that $\widetilde{f}$ has a conformal fixed point on the corresponding frontier component of A_i that agrees with the corresponding boundary circle of N. But $\widetilde{f}$ also has a conformal fixed point on the other frontier component of A_i, the one that is contained in $Q \cup Q_1$. This easily follows from Proposition 3.7 and the assumption that Q is a subset of $\mathcal{I}(K, \widetilde{f}, A)$. So we have that $\widetilde{f}$ has conformal fixed points on both frontier components of A_i, $i = 0, 1$. Therefore we have that $\rho(\widetilde{f}, A_i)$ is an integer. Since $A_1 = \widetilde{e}(A_0)$ we have that $\rho(\widetilde{f}, A_0) = \rho(\widetilde{f}, A_1)$ (because $\widetilde{f}$ commutes with $\widetilde{e}$). If $\operatorname{int}(B)$ (the interior of B) does not contain an annulus homotopic to N then by Lemma 3.2 we conclude that $\rho(\widetilde{f}, N) = \rho(\widetilde{f}, A_0) + \rho(\widetilde{f}, A_1) = 2\rho(\widetilde{f}, A_0)$, that is $\rho(\widetilde{f}, N)$ is an even integer which is a contradiction. So $\operatorname{int}(B)$ contains an annulus homotopic to N. Denote this annulus by D'. Let D'' be the union of D' and every connected component of the set $N \setminus \overline{D'}$ whose boundary is contained in $\overline{D'}$. Set $D = \operatorname{int}(D'')$. Then D is a faithful domain and $\partial D \subset B'$. Moreover the annuli A_0, A_1 and D are mutually disjoint. Again by Lemma 3.2 we have that $\rho(\widetilde{f}, N) = \rho(\widetilde{f}, A_0) + \rho(\widetilde{f}, A_1) + \rho(\widetilde{f}, D) = 2\rho(\widetilde{f}, A_0) + \rho(\widetilde{f}, D)$. This implies that $\rho(\widetilde{f}, D)$ is an odd integer. Since $D \subset A$ and $D \neq A$ we have found the required annulus which proves the proposition.

□

4. The minimal annulus

4.1. The groups $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$ and the characteristic annulus. From now on we assume that there exists a homomorphic section $\mathcal{E} : \mathcal{MC}(M) \to \operatorname{Homeo}(M)$, where M is a surface of genus $\mathbf{g} \geq 2$. By the end of the paper we will obtain a contradiction. From now on we fix the complex structure on M so that the homeomorphism $\mathcal{E}(e)$ is a conformal involution (it is well known that such a structure exists).

Recall from Section 2.1 the definition of curves α_i, β_j, and γ (and γ_1 in case when $\mathbf{g}$ is odd). For a fixed pair $(i, j) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ let $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$ be the group generated by the homeomorphisms $\mathcal{E}(t_{\alpha_k})$, $\mathcal{E}(t_{\beta_l})$, where $\alpha_k \in \mathbf{a}_i$ and $\beta_l \in \mathbf{b}_j$. Let $\mathbf{S}(i, j)$ denote the minimal decomposition of M for the group $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$, and let $M_{\mathbf{S}(i,j)}$ be the union all acyclic components of $\mathbf{S}(i, j)$. By $\mathbf{S}(\gamma)$ we denote the minimal decomposition for the homeomorphism $\mathcal{E}(t_\gamma)$, and by $M_{\mathbf{S}(\gamma)}$ we denote the union of all acyclic components in $\mathbf{S}(\gamma)$.

It follows from Lemma 2.1 that if γ is a separating curve (that is $\mathbf{g}$ is even) then there exist precisely two non-planar components of $M_{\mathbf{S}(\gamma)}$. Each component has exactly one end and this end is homotopic to γ. Let M_γ be the union of these two components. If γ is non-separating, then $M_{\mathbf{S}(\gamma)}$ has precisely one non-planar component, and this component has two ends that are both homotopic to γ. Let M_γ be this non-planar component in this case.

Then $\mathcal{E}(t_\gamma)$ setwise preserves M_γ. Since the involution e commutes with t_γ we have that $\mathcal{E}(e)$ setwise preserves M_γ. If $\mathbf{g}$ is even then $\mathcal{E}(e)$ exchanges the two components of M_γ and if $\mathbf{g}$ is odd then $\mathcal{E}(e)$ exchanges the two ends of M_γ. Set $B_\gamma = M \setminus M_\gamma$. In both cases ($\mathbf{g}$ odd or even) we have that B_γ is a closed, connected and non-acyclic subset of M, and $\mathcal{E}(t_\gamma)(B_\gamma) = \mathcal{E}(e)(B_\gamma) = B_\gamma$.

Let $\gamma' \subset M_\gamma$ and $\gamma'' = \mathcal{E}(e)(\gamma')$ be two simple closed curves on M representing the two ends of $M_\gamma = M \setminus B_\gamma$. Then $M_\gamma \setminus \{\gamma', \gamma''\}$ has either three or four components depending whether M_γ is connected or not. We denote by $M_\gamma(\gamma', \gamma'')$ either the component, or the union of two components of $M_\gamma \setminus \{\gamma', \gamma''\}$ whose boundary consists only of γ' and γ''. Then

$$\overline{M_\gamma(\gamma', \gamma'')} = M_\gamma(\gamma', \gamma'') \cup \gamma' \cup \gamma''$$

is the closure of $M_\gamma(\gamma', \gamma'')$. Note that $\mathcal{E}(e)$ setwise preserves $M_\gamma(\gamma', \gamma'')$ (but $\mathcal{E}(t_\gamma)$ does not necessarily preserve this set).

Let $C_\gamma(\gamma', \gamma'')$ denote the union of all acyclic components of $\mathbf{S}(\gamma)$ which intersect $\overline{M_\gamma(\gamma', \gamma'')}$. The set $C_\gamma(\gamma', \gamma'')$ is a compact subset of M_γ which is either connected or it has two components depending whether γ is a non-separating or a separating curve. Note that $\mathcal{E}(t_\gamma)$ setwise preserves the set $C_\gamma(\gamma', \gamma'')$ (because this set is a union of components from $\mathbf{S}(\gamma)$ and these by definition are setwise preserved by $\mathcal{E}(t_\gamma)$). Also, $\mathcal{E}(e)$ setwise preserves $C_\gamma(\gamma', \gamma'')$. This follows from the fact that $\mathcal{E}(e)$ setwise preserves $M_\gamma(\gamma', \gamma'')$, and since the homeomorphisms $\mathcal{E}(e)$ and $\mathcal{E}(t_\gamma)$ commute, we have that if $S \in M_{\mathbf{S}(\gamma)}$ then $\mathcal{E}(e)(S) \in M_{\mathbf{S}(\gamma)}$ as well.

Let A_{ch} denote the component of the complement of $C_\gamma(\gamma', \gamma'')$ that contains B_γ. Then A_{ch} is an open topological annulus (or just annulus) homotopic to γ. The annulus A_{ch} is called the characteristic annulus (see [7]). We have

$$\mathcal{E}(t_\gamma)(A_{\mathrm{ch}}) = \mathcal{E}(e)(A_{\mathrm{ch}}) = A_{\mathrm{ch}}. \tag{11}$$

Let $\Phi : N \to A_{\mathrm{ch}}$ be a conformal map to the corresponding geometric annulus N. By [7, Lemma 5.1], the map $\Phi^{-1} \circ \mathcal{E}(t_\gamma) \circ \Phi : \overline{N} \to \overline{N}$ has at least one fixed point on each boundary component of $\overline{N}$ and it is homotopic (modulo its fixed points on the boundary) to the standard twist homeomorphism.

The annulus A_{ch} depends on the choice of γ' and γ'' but this is not relevant in this paper (that is we will not be changing the choice of these two curves). From now on A_{ch} is fixed and we will return to it later.

Lemma 4.1. *Fix $i, j \in \{1, 2\}$. The set $M_{\mathbf{S}(i,j)}$ contains a unique connected component $M_{\mathbf{S}(i,j)}(\gamma)$ with the following properties*

- *$M_{\mathbf{S}(i,j)}(\gamma)$ has negative Euler characteristic.*
- *$M_{\mathbf{S}(i,j)}(\gamma)$ contains a curve homotopic to γ.*
- *No end of $M_{\mathbf{S}(i,j)}(\gamma)$ is homotopic to γ and every end of $M_{\mathbf{S}(i,j)}(\gamma)$ is essential in M (this means that every end of $M_{\mathbf{S}(i,j)}(\gamma)$ is homotopic to a simple closed curve in M that is not homotopically trivial).*

Remark. The homeomorphism $\mathcal{E}(e)$ permutes the sets $M_{\mathbf{S}(i,j)}(\gamma)$ that is $\mathcal{E}(e)(M_{\mathbf{S}(i,j)}(\gamma)) = M_{\mathbf{S}(j,i)}(\gamma)$.

Proof. Observe that we may choose the curves in $\mathbf{a}_i$ and $\mathbf{b}_j$ such that every two curves from $\mathbf{a}_i \cup \mathbf{b}_j$ are mutually disjoint and any curve from $\mathbf{a}_i \cup \mathbf{b}_j$ is disjoint from the curve γ (see Figure 1, Figure 2, Figure 3). Then the set $M \setminus (\mathbf{a}_i \cup \mathbf{b}_j)$ contains a component C of negative Euler characteristic such that $\gamma \subset C$ and that no end of C is homotopic to γ. Moreover every end of C is essential in M. By Lemma 2.1 we conclude that $M_{\mathbf{S}(i,j)}$ has a component $M_{\mathbf{S}(i,j)}(\gamma)$ homotopic to C. $\square$

Proposition 4.1. *Let $A \subset M$ be an annulus with the following properties*

(1) *The annulus A is homotopic to γ.*
(2) $\mathcal{E}(t_\gamma)(A) = A$.
(3) *The twist number $\rho(\mathcal{E}(t_\gamma), A)$ is an odd integer.*

Let $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$ be one of the four groups we defined above. Then there exists a unique connected component A_1 of the set $A \cap M_{\mathbf{S}(i,j)}(\gamma)$ such that that A_1 is a topological annulus homotopic to γ. Moreover A_1 satisfies the properties (1), (2) *and* (3). *In fact we have $\rho(\mathcal{E}(t_\gamma), A) = \rho(\mathcal{E}(t_\gamma), A_1)$.*

Proof. We first study the set $A \cap (M \setminus M_{\mathbf{S}(i,j)}(\gamma))$. Since $M_{\mathbf{S}(i,j)}(\gamma)$ contains a curve homotopic to γ and γ is not homotopic to any end of $M_{\mathbf{S}(i,j)}(\gamma)$, we conclude that the set $M \setminus M_{\mathbf{S}(i,j)}(\gamma)$ does not contain a curve homotopic to γ. For $\epsilon > 0$ let B_ϵ denote an ϵ-neighbourhood (with respect to the hyperbolic metric on M that we fixed above) of the set $M \setminus M_{\mathbf{S}(i,j)}(\gamma)$. Then we may choose ϵ small enough so that the set B_ϵ does not contain a curve homotopic to γ. Fix such $\epsilon > 0$. Then no connected component of the set $A \cap B_\epsilon$ can separate the two frontier components of A. Now we show that no connected component of the set $A \cap B_\epsilon$ can connect the two frontier components of A.

Let $0 < \delta < \epsilon$ be such that
$$\mathbf{d}((\mathcal{E}(t_\gamma))^{10}(x), (\mathcal{E}(t_\gamma))^{10}(y)) < \epsilon,$$
whenever $x, y \in M$ and $\mathbf{d}(x,y) < \delta$. This implies $(\mathcal{E}(t_\gamma))^{10}(B_\delta) \subset B_\epsilon$. Assume that there exists a connected component of the set $A \cap B_\delta$ that connects the two frontier components of A. Then there exists a Jordan arc $l \subset \overline{A} \cap B_\delta$ that connects the two boundary components $\partial_0(A)$ and $\partial_1(A)$ of A. Since $\mathcal{E}(t_\gamma)$ has an odd rotation number and it fixes A, it follows from (4) that $|\iota([l], [\mathcal{E}(t_\gamma)^{10}(l)])| \geq 2$. This implies that $\big((\mathcal{E}(t_\gamma))^{10}(l) \cup l\big)$ separates the two frontier components of A. This is a contradiction since $\big((\mathcal{E}(t_\gamma))^{10}(l) \cup l\big)$ is contained in B_ϵ. Therefore no connected component of the set $A \cap B_\epsilon$ can connect the two ends of A.

We have that no component of the set $A \cap (M \setminus M_{\mathbf{S}(i,j)}(\gamma))$ can separate or connect the two frontier components of A. Also the sets
$$\partial_0(A) \setminus \big(M \setminus M_{\mathbf{S}(i,j)}(\gamma)\big), \quad \text{and} \quad \partial_1(A) \setminus \big(M \setminus M_{\mathbf{S}(i,j)}(\gamma)\big),$$
are non-empty and relatively open. Combining this with the fact that $M_{\mathbf{S}(i,j)}(\gamma)$ is connected we conclude that the set $A \cap M_{\mathbf{S}(i,j)}(\gamma)$ contains a component that is an annulus homotopic to γ. Since $M_{\mathbf{S}(i,j)}(\gamma)$ is connected such annulus is unique. Denote this annulus by A_1. We show that A_1 satisfies the same properties as A.

We have already seen that A_1 is homotopic to γ. Since $\mathcal{E}(t_\gamma)$ setwise preserves $M_{\mathbf{S}(i,j)}(\gamma)$ we conclude that $\mathcal{E}(t_\gamma)$ setwise preserves A_1. Since the sets $\partial_0(A) \setminus (M \setminus$

$M_{\mathbf{S}(i,j)}(\gamma))$ and $\partial_1(A) \setminus (M \setminus M_{\mathbf{S}(i,j)}(\gamma))$ are non-empty there exists a Jordan arc $l \subset A_1$ that connects the two frontier components components of A. It follows from Proposition 3.4 that $\rho(\mathcal{E}(t_\gamma), A) = \rho(\mathcal{E}(t_\gamma), A_1)$. □

4.2. **The minimal annulus.** We have

Definition 4.1. *Let A_{ch} be the characteristic annulus defined above. Let A be a topological annulus on M homotopic to γ. The annulus A is said to be an admissible annulus if it satisfies the following*

(1) $A \subset A_{\text{ch}}$.
(2) $\mathcal{E}(e)(A) = A$
(3) $\mathcal{E}(t_\gamma)(A) = A$
(4) *The twist number $\rho(\mathcal{E}(t_\gamma), A)$ is an odd integer.*
(5) $A \subset M_{\mathbf{S}(i,j)}(\gamma)$ *for any pair $i, j \in \{1, 2\}$.*

A topological annulus A is said to be a minimal annulus if it is admissible and if no other admissible annulus is strictly contained in A.

Proposition 4.2. *There exists an admissible annulus.*

Remark. In fact we have that $\rho(\mathcal{E}(t_\gamma), A) = 1$ for the admissible annulus we construct in the proof below..

Proof. Consider the set

$$B = A_{\text{ch}} \bigcap \left(\bigcap_{i,j \in \{1,2\}} M_{\mathbf{S}(i,j)}(\gamma) \right).$$

By Proposition 4.1 there exists a unique connected component $A(1,1)$ of $A_{\text{ch}} \cap M_{\mathbf{S}(1,1)}(\gamma)$ that is an annulus homotopic to γ. Moreover $\mathcal{E}(t_\gamma)$ setwise preserves this annulus and $\rho(\mathcal{E}(t_\gamma), A(1,1)) = \rho(\mathcal{E}(t_\gamma), A_{\text{ch}}) = 1$. That is the annulus $A(1,1)$ satisfies the assumptions of Proposition 4.1. We apply this proposition again and find that $A(1,1) \cap M_{\mathbf{S}(1,2)}(\gamma)$ contains a unique component $A(1,2)$ that is an annulus homotopic to γ. Again $A(1,2)$ satisfies the assumptions of Proposition 4.1. We repeat this two more times. We conclude that the set B has a component A that is an annulus homotopic to γ, such that $\mathcal{E}(t_\gamma)(A) = A$, and such that $\rho(\mathcal{E}(t_\gamma), A) = 1$.

It remains to show that $\mathcal{E}(e)(A) = A$. Set $A_1 = \mathcal{E}(e)(A)$. It follows from (11) (also see Lemma 4.1) that $\mathcal{E}(e)$ setwise preserves the set B. We conclude that A_1 is also a connected component of the set B. In particular we have that either $A = A_1$ or $A \cap A_1 = \emptyset$, and both annuli A and A_1 are homotopic to γ. We show that $A = A_1$. If $A \cap A_1 = \emptyset$ then the set $M \setminus (A \cup A_1)$ contains a unique component B' such that $A \cup A_1 \cup B' = A'$ is an annulus homotopic to γ. Moreover, the boundary of A' is a subset of $\partial A \cup \partial A_1$. Since each subsurface $M_{\mathbf{S}(i,j)}(\gamma)$ contains $A \cup A_1$ and since the ends of $M_{\mathbf{S}(i,j)}(\gamma)$ are essential (see Lemma 4.1) we conclude that $A' \subset M_{\mathbf{S}(i,j)}(\gamma)$ for each i, j. This shows that $A = A_1$. □

Proposition 4.3. *There exists a minimal annulus.*

Remark. It can be shown that in fact there exists a unique minimal annulus. We do not need this result so we omit proving it.

Proof. Consider the family $\mathcal{F}$ of all admissible annuli. This family is non-empty by the previous proposition. The partial ordering on $\mathcal{F}$ is given by the inclusion.

By the Zorn's lemma there exists a maximal chain $\mathcal{A}$ in $\mathcal{F}$. We show that the intersection of all annuli in $\mathcal{A}$ is a set whose interior contains an annulus that also belongs to $\mathcal{A}$ (that is we show that $\mathcal{A}$ has the minimal element). This annulus is then by definition a minimal annulus.

Since M is a separable space, it follows that there exists a decreasing sequence of annuli $A_n \in \mathcal{C}$ such that

$$\bigcap_{n\in\mathbf{N}} A_n = \bigcap_{A\in\mathcal{A}} A.$$

Let $\mathbf{d}$ denote the corresponding hyperbolic distance on M (recall that we have fixed the complex structure on M). Let $D \subset A_n$ be a geodesic disc (with respect to the hyperbolic metric). We say that D is a proper maximal disc if the closed disc $\overline{D}$ has non-empty intersection with both frontier components of A_n.

Remark. If $\Omega \subset M$ is a domain we say that $D \subset \Omega$ is a maximal disc in Ω if D is a geodesic disc that is not contained in any larger hyperbolic disc which is a subset of Ω. Then the closed disc $\overline{D}$ has to touch the boundary of Ω. If Ω is a topological annulus then a closed maximal disc does not need to connect the two frontier components of A_n. This is why we call such discs proper maximal discs.

If $z \in \partial A_n$ is a point where $\overline{D}$ touches the boundary ∂A_n then z is an accessible point, and the geodesic arc that connects the centre of D with z is contained in A. Let D_n be a proper maximal disc that has the smallest radius among all proper maximal discs in A_n (there could be more than one such disc with the smallest radius and we pick one). Let r_n denote the radius of D_n and let c_n be its centre. We show that $\liminf\limits_{n\to\infty} r_n > 0$.

Since D_n is a proper maximal disc there exist points $z_n \in \partial_0(A_n)$ and $w_n \in \partial_1(A_n)$ that are in the closed disc $\overline{D}_n$. Let $l_n \subset D_n \subset A_n$ be the arc that has the endpoints z_n and w_n, such that l_n is the union of the two geodesic arcs that connect z_n and w_n with c_n respectively. Let $l'_n = (\mathcal{E}(t_\gamma))^{10}(l_n)$. Since $\rho(\mathcal{E}(t_\gamma), A_n)$ is odd, it follows from Proposition 3.4 (and formula (4)) that $|\iota([l_n],[l'_n])| \geq 2$. This implies that the set $l_n \cup l'_n$ contains a closed curve homotopic to γ. Denote this curve by t_n. For every $n \in \mathbf{N}$ the hyperbolic diameter of t_n is bounded below by the half of the hyperbolic length of the simple closed geodesic homotopic to γ.

Assume that after passing onto a subsequence if necessary, we have $r_n \to 0$, $n \to \infty$. Then the hyperbolic diameter of the arc l_n tends to zero. Since $\mathcal{E}(t_\gamma)^{10}$ is uniformly continuous on M we see that the hyperbolic diameter of l'_n tends to zero as well. Since $t_n \subset (l_n \cup l'_n)$ we conclude that the hyperbolic diameter of t_n tends to zero. But this is a contradiction with the fact that the the hyperbolic diameter of t_n is bounded away from zero.

Let $1 < s_n$ so that

$$\mathrm{Mod}(A_n) = \frac{\log s_n}{2\pi}.$$

Here $\mathrm{Mod}(A_n)$ denotes the conformal modulus of A_n. Then there exists a conformal map $\Phi_n : N(s_n) \to M$ such that $\Phi_n(N(s_n)) = A_n$. Since the radius of every proper maximal disc is bounded away from zero (regardless of n) we conclude that the distance between the two frontier components of A_n is bounded away from zero (regardless of n). Combining this with the fact that the sequence A_n is decreasing we have that $\lim\limits_{n\to\infty} s_n = s$ exists and $1 < s$ (see [6]). This shows that the sequence of conformal maps Φ_n converges on every compact set in $N(s)$ (after passing onto a subsequence if necessary) to a non-degenerate conformal map $\Phi : N(s) \to M$ (note that every compact set in $N(s)$ is eventually contained in $N(s_n)$ which is the domain of Φ_n). Let $A = \Phi(N(s))$. Then A is a topological annulus that is contained in $\bigcap_{n\in\mathbf{N}} A_n$. Clearly A is setwise preserved by $\mathcal{E}(t_\gamma)$ and $\mathcal{E}(e)$. It remains to show that the twist number $\rho(\mathcal{E}(t_\gamma), A)$ is an odd integer.

Going back to the sequence of proper maximal discs D_n we see that after passing to a subsequence if necessary, we have that $D_n \to D$ where D is a proper maximal discs in A. Also the sequence of arcs l_n converges to the corresponding arc $l \subset A$ whose endpoints are in the opposite frontier components of A (recall that each l_n constitutes of the two geodesic arcs and so does l). Fix $k \in \mathbf{N}$. Then by the continuity of $(\mathcal{E}(t_\gamma))^k$, for n large enough we have that

$$|\iota([l_n], [(\mathcal{E}(t_\gamma))^k(l_n)]) - \iota([l], [(\mathcal{E}(t_\gamma))^k(l)])| \leq 2.$$

Remark. If $(\mathcal{E}(t_\gamma))^k$ does not fix either endpoint of l then for n large enough we have that $\iota([l_n], [(\mathcal{E}(t_\gamma))^k(l_n)]) = \iota([l], [(\mathcal{E}(t_\gamma))^k(l)])$. But if $(\mathcal{E}(t_\gamma))^k$ fixes one or both endpoints of l then the two numbers may differ by 2.

By the previous inequality and from Proposition 3.4 (formula (4)) we have

$$|k\rho((\mathcal{E}(t_\gamma)), A_n) - k\rho((\mathcal{E}(t_\gamma)), A)| \leq 20,$$

for every $k \in \mathbf{N}$ and n large enough. We have

$$\lim_{n\to\infty} \rho(\mathcal{E}(t_\gamma), A_n) = \rho(\mathcal{E}(t_\gamma), A).$$

Since every $\rho(\mathcal{E}(t_\gamma), A_n)$ is an odd integer so is $\rho(\mathcal{E}(t_\gamma), A)$.

□

4.3. **The action of the twist $\mathcal{E}(t_\gamma)$ on the minimal annulus.** From now on we fix a minimal annulus and call it $A_{\min}$. We may assume that the strip P is the universal cover of $A_{\min}$. Let $p \in A_{\min}$ and let $S_p(i,j) \in \mathbf{S}(i,j)$ be the corresponding component that contains p (here $i, j \in \{1,2\}$). Let $\widehat{S}_p(i,j)$ denote a single lift of $S_p(i,j)$ to P. Since $S_p(i,j)$ is acyclic we have that every connected component of the set $P \cap \widehat{S}_p(i,j)$ has a finite Euclidean diameter (if a relatively closed subset of P has an infinite diameter, and if this set is invariant for the translation for 1, then the projection of this set to $A_{\min}$ is not acyclic). Let C be the supremum of such Euclidean diameters when $p \in A_{\min}$ and $i, j \in \{1,2\}$. Since $\mathbf{S}(i,j)$ is upper semi-continuous we find that this supremum is achieved and we denote it by $C_{\min}$.

Proposition 4.4. *Let $S \in \mathbf{S}(i,j)$, $i, j \in \{1,2\}$, and assume that S has a non-empty intersection with $A_{\min}$. Then S is acyclic and S is compactly contained in $A_{\min}$.*

Proof. It follows from the definition of $A_{\min}$ that for each point $p \in A_{\min}$, the corresponding component $S_p(i,j) \in \mathbf{S}(i,j)$ that contains p is acyclic. Since $\mathcal{E}(t_\gamma)$ commutes with the elements of $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$, it follows that $\mathcal{E}(t_\gamma)$ permutes the components of the minimal decompositions $\mathbf{S}(i,j)$, that is $\mathcal{E}(t_\gamma)(S_p(i,j)) = S_{\mathcal{E}(t_\gamma)(p)}(i,j)$. First we show that $S_p(i,j)$ cannot connect the two frontier components of $A_{\min}$.

Assume on the contrary that $S_p(i,j)$ connects the two frontier components of $A_{\min}$. Let S be a connected component of $S_p(i,j) \cap A_{\min}$ such that $\overline{S}$ connects the two frontier components. Let $\widehat{S}$ be a lift of S to P. Then the closure $\overline{\widehat{S}}$ of $\widehat{S}$ connects the two boundary lines of P. Let $z_i \in \partial_i(P) \cap \overline{\widehat{S}}$, $i = 0, 1$. Let $\widehat{f}$ be a lift of $\mathcal{E}(t_\gamma)$ to P. Then $|\widehat{f}^n(z_1)-\widehat{f}^n(z_2)|\le C_{\min}$, for every $n\in\mathbf{Z}$. By Proposition 3.2 we have that $\rho(\mathcal{E}(t_\gamma),A_{\min})=0$.

Next we show that $S_p(i,j)$ cannot intersect the boundary of $A_{\min}$. Assume on the contrary that $S_p(i,j)$ intersects the frontier component $\partial_0(A_{\min})$. We already showed that $S_p(i,j)$ cannot connect the two frontier components of $A_{\min}$. Since $\mathcal{E}(t_\gamma)$ preserves each frontier component of $A_{\min}$, it follows that the set

$$X=\bigcup_{k\in\mathbb{Z}}\mathcal{E}(t_\gamma)^k(S_p(i,j)\bigcap\overline{A_{\min}}),$$

intersects only $\partial_0(A_{\min})$. Moreover the closure $\overline{X}$ cannot connect the two frontier components of $A_{\min}$ either. If we assume that $\overline{X}$ connects the two frontier components of $A_{\min}$ then by the upper semi-continuity of the decomposition $\mathbf{S}(i,j)$ there exists a single component of $\mathbf{S}(i,j)$ connecting the two frontier components of $A_{\min}$, which is a contradiction.

We also claim that $\overline{X}\cap\mathcal{E}(e)(\overline{X})=\emptyset$. Note that $\mathcal{E}(e)(\mathbf{S}(i,j))=\mathbf{S}(j,i)$. Assume that $\overline{X}\cap\mathcal{E}(e)(\overline{X})\neq\emptyset$ and let $p\in\overline{X}\cap\mathcal{E}(e)(\overline{X})$. Then $S_p(i,j)\cup S_p(j,i)$ is a closed set that connects the two frontier components of $A_{\min}$. Let S_1 and S_2 be connected components of the sets $S_p(i,j)\cap A_{\min}$ and $S_p(j,i)\cap A_{\min}$ respectively, such that $S_1\cup S_2$ connects the two frontier components of $A_{\min}$. Let $\widehat{S}_1$ and $\widehat{S}_2$ be the corresponding single lifts to P such that $\widehat{S}_1\cup\widehat{S}_2$ connects the two boundary lines of P. Let $z_0\in\partial_0(P)\cap\overline{(\widehat{S}_1\cup\widehat{S}_2)}$ and $z_1\in\partial_1(P)\cap\overline{(\widehat{S}_1\cup\widehat{S}_2)}$. Let $\widehat{f}$ be a lift of $\mathcal{E}(t_\gamma)$ to P. Then the Euclidean diameter of the set $\widehat{f}^n(\widehat{S}_1\cup\widehat{S}_2)$ is less than $2C_{\min}$, where $C_{\min}$ is the constant defined above. This shows that $|\widehat{f}^n(z_1)-\widehat{f}^n(z_0)|\le 2C_{\min}$, for every $n\in\mathbf{Z}$. But this contradicts Proposition 3.2 since $\rho(\mathcal{E}(t_\gamma),A_{\min})\neq 0$.

Since $\overline{X}\cap\mathcal{E}(e)(\overline{X})=\emptyset$ we have that the set $A_{\min}\setminus(\overline{X}\cup\mathcal{E}(e)(\overline{X}))$ contains a unique component A that is a topological annulus which is invariant under both $\mathcal{E}(t_\gamma)$ and $\mathcal{E}(e)$. We will show that A is an admissible annulus and will contradict that $A_{\min}$ is a minimal annulus.

To show that A is admissible it remains to show that the rotation number of $\mathcal{E}(t_\gamma)$ on A is an odd integer. There are two cases to consider. The first one is when $\partial A\cap\partial A_{\min}\neq\emptyset$. Since $A=\mathcal{E}(e)(A)$ we have that there are points $z_i\in\partial A\cap\partial_i(A_{\min})$. If z_i are accessible points (with respect to A) we can choose an arc $l\subset A$ that connects the two points. Then by Proposition 3.4 we have that $\rho(\mathcal{E}(t_\gamma),A_{\min})=\rho(\mathcal{E}(t_\gamma),A)$. If z_i is not accessible we can find accessible points that are arbitrary close to z_i and then the argument goes the same way as in the proof of Lemma 3.2. This shows that $\rho(\mathcal{E}(t_\gamma),A_{\min})=\rho(\mathcal{E}(t_\gamma),A)$.

If $\partial A\cap\partial A_{\min}=\emptyset$ then there are two mutually disjoint annuli $A_i\subset A_{\min}$, $i=0,1$ (and disjoint from A) such that $\partial_i(A_i)=\partial_i(A_{\min})$. Moreover we have that $S_p(i,j)$ connects the two frontier components of A_0 and $S_p(j,i)$ connects the two frontier components of A_1. Same as above we show that $\rho(\mathcal{E}(t_\gamma),A_0)=\rho(\mathcal{E}(t_\gamma),A_1)=0$. From Lemma 3.2 we have $\rho(\mathcal{E}(t_\gamma),A_{\min})=\rho(\mathcal{E}(t_\gamma),A)+\rho(\mathcal{E}(t_\gamma),A_0)+\rho(\mathcal{E}(t_\gamma),A_1)=\rho(\mathcal{E}(t_\gamma),A)$. This proves that A is admissible. Therefore $S_p(i,j)$ does not intersect the boundary of $A_{\min}$ and since $S_p(i,j)$ is a closed set this proves that it is compactly contained in $A_{\min}$. $\square$

Proposition 4.5. *Let $p\in A_{\min}$ and $i,j\in\{1,2\}$. Then a single lift of $S_p(i,j)$ to P has the Euclidean diameter at most $C_{\min}$ ($C_{\min}$ is the constant defined above).*

Proof. This follows directly from the definition of $C_{\min}$ and the previous proposition. $\square$

Proposition 4.6. *Let $K > 0$. Then every connected component of the set $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ is compactly contained in $A_{\min}$ and it does not separate the two frontier components of $A_{\min}$.*

Proof. The proof of the statement that every connected componet of the set $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ is compactly contained in $A_{\min}$ is the same as the proof of Proposition 4.4 (in fact we have already proved in Proposition 3.9 that a connected component of $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ can not connect the two frontier components of $A_{\min}$). If Q is a connected component of $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ then we treat the set $\overline{Q} \subset \overline{A}_{\min}$ in the same way as the component $S_p(i,j)$ in the above proof.

Since $A_{\min}$ is a minimal annulus (and as such it does contain another admissible annulus) we conclude from Proposition 3.10 that a connected component of $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ can not separate the two frontier components of $A_{\min}$. $\square$

Proposition 4.7. *Let $K > 0$. Then there exists a non-empty connected component $D_{\min}$ of the set $A_{\min} \setminus \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ such that every connected component of the set $A_{\min} \setminus D_{\min}$ is compactly contained in $A_{\min}$.*

Remark. Observe that if Q is connected component of the set $A_{\min} \setminus D_{\min}$ then beside being compactly contained in $A_{\min}$ we have that Q does not separate the two frontier components of $A_{\min}$ because $D_{\min}$ is connected. Let $\widehat{D}_{\min}$ be the lift of $D_{\min}$ to P under the covering map. Then $\widehat{D}_{\min}$ is connected. Moreover every connected component of the set $P \setminus \widehat{D}_{\min}$ is compactly contained in P

Proof. By the previous propositions we know that every connected component of the set $\mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ is compactly contained in $A_{\min}$ and it does not separate the two frontier components of $A_{\min}$. Therefore we can find an arc

$$\gamma \subset A_{\min} \setminus \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min}),$$

such that γ connects the two frontier components of $A_{\min}$. Let $D_{\min}$ be the connected component of the set $A_{\min} \setminus \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ that contains γ. We need to show that every connected component of the set $A_{\min} \setminus D_{\min}$ is compactly contained in $A_{\min}$. Let Q' be such a component and let $Q = Q' \setminus \mathrm{int}(Q')$. Then $Q \subset \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$. If Q' is not compactly contained in $A_{\min}$ then neither is Q. But this is impossible. This proves the proposition. $\square$

Remark. In the previous proof we have that each connected component Q' of the set $A_{\min} \setminus D_{\min}$ is contained in some $\mathcal{I}(K', \mathcal{E}(t_\gamma), A_{\min})$, where K' depends on Q'.

Proposition 4.8. *Let $\widetilde{a}_i \in \Gamma(\mathbf{a}_i)$ and $\widetilde{b}_j \in \Gamma(\mathbf{b}_j)$. Let $\widehat{a}_i, \widehat{b}_j : \overline{P} \to \overline{P}$ be two lifts. Then there exist $k, l \in \mathbf{Z}$, and a map $\chi : P \to P$ such that $\chi \circ \widehat{a}_i = T^k \circ \chi$, and $\chi \circ \widehat{b}_j = T^l \circ \chi$, where T^k and T^l are translations for k and l respectively. Moreover, if $\widehat{a}_i$ has a fixed point in P then $k = 0$. Similarly if $\widehat{b}_j$ has a fixed point in P then $l = 0$.*

Proof. Let $\widehat{\mathbf{S}}(\mathbf{a}_i, \mathbf{b}_j)$ be the lift of the minimal decomposition for the group $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$ to P. Let $\chi : P \to P$ be the Moore's map, that is χ maps every component of $\widehat{\mathbf{S}}(\mathbf{a}_i, \mathbf{b}_j)$ to a point. Then χ is the required map. If $\widehat{a}_i(p) = p$, for some $p \in P$, then $\chi(p) = T^k(\chi(p))$ which shows that $k = 0$. The same argument goes if $\widehat{b}_j$ has a fixed point in P. □

5. The proof of Theorem 1.1

5.1. **Special subsets of the minimal annulus.** So far we have considered the minimal decompositions $\mathbf{S}(i,j)$ for the groups $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$, $i, j \in \{1,2\}$. Let $\Gamma(\mathbf{a}_i)$, $i = 1, 2$, be the group genererated by all $\mathcal{E}(t_\alpha)$, where $\alpha \in \mathbf{a}_i$. Let $\Gamma(\mathbf{b}_i)$, $i = 1, 2$, be the group genererated by all $\mathcal{E}(t_\beta)$, where $\beta \in \mathbf{b}_i$. Let $\mathbf{S}(\mathbf{a}_i)$ and $\mathbf{S}(\mathbf{b}_j)$ be the corresponding minimal decompositions. Then for $p \in A_{\min}$ we have that $S_p(\mathbf{a}_i) \subset S_p(i,j)$, and $S_p(\mathbf{b}_j) \subset S_p(i,j)$. This implies that every such component $S_p(\mathbf{a}_i)$ (or $S_p(\mathbf{b}_j)$) is compactly contained in $A_{\min}$ and a single lift of $S_p(\mathbf{a}_i)$ (or $S_p(\mathbf{b}_j)$) to the strip P has the Euclidean diameter less than $C_{\min}$.

Definition 5.1. *Let $p \in A_{\min}$. We say that $p \in O_0$ if for some pair $i, j \in \{1,2\}$, we have that p belongs to the interior of the component $S_p(i,j) \in \mathbf{S}(i,j)$. We say that $p \in E$ if $\mathcal{E}(t_\gamma)$ setwise fixes at least one of the four components $S_p(\mathbf{a}_1) \in \mathbf{S}(\mathbf{a}_1)$, $S_p(\mathbf{a}_2) \in \mathbf{S}(\mathbf{a}_2)$, $S_p(\mathbf{b}_1) \in \mathbf{S}(\mathbf{b}_1)$, $S_p(\mathbf{b}_2) \in \mathbf{S}(\mathbf{b}_2)$. Set $X = A_{\min} \setminus (O_0 \cup E)$.*

Definition 5.2. *Let $i, j \in \{1,2\}$ be fixed. Define $X_{\mathbf{a}_i,\mathbf{b}_j}$ to be the set of all $p \in X$ such that every element of the group $\Gamma(\mathbf{a}_i)$ setwise fixes the component $S_p(\mathbf{b}_j) \in \mathbf{S}(\mathbf{b}_j)$. Define $X_{\mathbf{b}_i,\mathbf{a}_j}$ to be the set of all $p \in X$ such that every element of the group $\Gamma(\mathbf{b}_i)$ setwise fixes the component $S_p(\mathbf{a}_j) \in \mathbf{S}(\mathbf{a}_j)$.*

Proposition 5.1. *For every pair $i, j \in \{1,2\}$ we have*

$$X = X_{\mathbf{a}_i,\mathbf{b}_j} \bigcup X_{\mathbf{b}_j,\mathbf{a}_i}. \tag{12}$$

Proof. If $p \in X$ then p does not belong to the set O_0. The identity (12) then follows directly from Lemma 2.2. □

Now we use the Artin type relations introduced in Section 2.

Proposition 5.2. *We have $X_{\mathbf{a}_1,\mathbf{b}_i} \cap X_{\mathbf{a}_2,\mathbf{b}_i} = \emptyset$ and $X_{\mathbf{b}_1,\mathbf{a}_i} \cap X_{\mathbf{b}_2,\mathbf{a}_i} = \emptyset$, for every $i \in \{1,2\}$.*

Proof. We show $X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{a}_2,\mathbf{b}_1} = \emptyset$. The other case is proved in the same way. Assume that $X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{a}_2,\mathbf{b}_1} \neq \emptyset$. Let $p \in X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{a}_2,\mathbf{b}_1}$. Then all elements of both groups $\Gamma(\mathbf{a}_1)$ and $\Gamma(\mathbf{a}_2)$ setwise fix the set $S_p(\mathbf{b}_1)$. We apply the Artin type relation (when the genus of M is even we apply (1) and when genus is odd we apply (2)) and obtain that $\mathcal{E}(t_\gamma)$ setwises fixes the set $S_p(\mathbf{b}_1)$. This shows that $p \in E$ which contradicts the assumption $p \in X$ since $X \cap E = \emptyset$. □

Proposition 5.3. *We have $X_{\mathbf{a}_i,\mathbf{b}_j} \cap X_{\mathbf{b}_j,\mathbf{a}_i} = \emptyset$ for every pair $i, j \in \{1,2\}$.*

Proof. Assume that $X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{b}_1,\mathbf{a}_1} \neq \emptyset$. We derive a contradiction. Let $p \in X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{b}_1,\mathbf{a}_2}$. Then every element of the group $\Gamma(\mathbf{a}_1)$ setwise fixes the component $S_p(\mathbf{b}_1) \in \mathbf{S}(\mathbf{b}_1)$ and every element of the group $\Gamma(\mathbf{b}_1)$ setwise fixes the component $S_p(\mathbf{a}_1) \in \mathbf{S}(\mathbf{a}_1)$. Now we apply (12) to the pair $(i,j) = (1,2)$. This shows that at least one of the following holds:

1. Every element of the group $\Gamma(\mathbf{a}_1)$ setwise fixes the component $S_p(\mathbf{b}_2)$.
2. Every element of the group $\Gamma(\mathbf{b}_2)$ setwise fixes the component $S_p(\mathbf{a}_1)$.

Assume that (2) holds. Then the conclusion is that all elements of both groups $\Gamma(\mathbf{b}_1)$ and $\Gamma(\mathbf{b}_2)$ setwise fix the set $S_p(\mathbf{a}_1)$. We then apply the Artin type relation (when the genus of M is even we apply (1) and when genus is odd we apply (2)) and obtain that $\mathcal{E}(t_\gamma)$ setwises fixes the component $S_p(\mathbf{a}_1)$. But then $p \in E$ which contradits that $p \in X$. This shows that the (2) can not hold so we conclude that every element of the group $\Gamma(\mathbf{a}_1)$ setwise fixes the component $S_p(\mathbf{b}_2)$. If we apply (12) to the pair $(i,j) = (2,1)$, by the same argument we conclude that every element of the group $\Gamma(\mathbf{b}_1)$ setwise fixes the component $S_p(\mathbf{a}_2)$. Let us collect the statements we have proved so far

1. Every element of the group $\Gamma(\mathbf{a}_1)$ setwise fixes the component $S_p(\mathbf{b}_1)$.
2. Every element of the group $\Gamma(\mathbf{b}_1)$ setwise fixes the component $S_p(\mathbf{a}_1)$.
3. Every element of the group $\Gamma(\mathbf{a}_1)$ setwise fixes the component $S_p(\mathbf{b}_2)$.
4. Every element of the group $\Gamma(\mathbf{b}_1)$ setwise fixes the component $S_p(\mathbf{a}_2)$.

We now apply (12) to the pair $(i,j) = (2,2)$. If every element of the group $\Gamma(\mathbf{a}_2)$ setwise fixes the component $S_p(\mathbf{b}_2)$ then we have that all elements of both groups $\Gamma(\mathbf{a}_1)$ and $\Gamma(\mathbf{a}_2)$ setwise fix the set $S_p(\mathbf{b}_2)$. Again by the corresponding Artin type relation this implies that $\mathcal{E}(t_\gamma)$ setwises fixes the component $S_p(\mathbf{b}_2)$. This shows that $p \in E$ which is a contradiction. Similarly if every element of the group $\Gamma(\mathbf{b}_2)$ setwise fixes the component $S_p(\mathbf{a}_2)$ then all elements of both groups $\Gamma(\mathbf{b}_1)$ and $\Gamma(\mathbf{b}_2)$ setwise fix the set $S_p(\mathbf{a}_2)$. Again a contradiction.

One similarly shows that $X_{\mathbf{a}_i,\mathbf{b}_j} \cap X_{\mathbf{b}_j,\mathbf{a}_i} = \emptyset$ for other three pairs (i,j). □

Lemma 5.1. *We have*

$$X_1 = X_{\mathbf{b}_1,\mathbf{a}_1} = X_{\mathbf{a}_1,\mathbf{b}_2} = X_{\mathbf{b}_2,\mathbf{a}_2} = X_{\mathbf{a}_2,\mathbf{b}_1},$$

and

$$X_2 = X_{\mathbf{b}_2,\mathbf{a}_1} = X_{\mathbf{a}_1,\mathbf{b}_1} = X_{\mathbf{b}_1,\mathbf{a}_2} = X_{\mathbf{a}_2,\mathbf{b}_2}.$$

The sets X_1 and X_2 are disjoint.

Proof. The first two identities follow from the previous three propositions. This is seen as follows. We have $X_{\mathbf{b}_1,\mathbf{a}_1} \cup X_{\mathbf{a}_1,\mathbf{b}_1} = X$ and this union is disjoint. Since $X_{\mathbf{b}_1,\mathbf{a}_1} \cap X_{\mathbf{b}_2,\mathbf{a}_1} = \emptyset$ we have

$$X_{\mathbf{b}_2,\mathbf{a}_1} \subset X_{\mathbf{a}_1,\mathbf{b}_1}.$$

Since $X_{\mathbf{a}_1,\mathbf{b}_1} \cap X_{\mathbf{a}_2,\mathbf{b}_1} = \emptyset$ we have

$$X_{\mathbf{a}_2,\mathbf{b}_1} \subset X_{\mathbf{b}_1,\mathbf{a}_1}.$$

Next, we have $X_{\mathbf{b}_2,\mathbf{a}_1} \cup X_{\mathbf{a}_1,\mathbf{b}_2} = X$ and this union is disjoint. Since $X_{\mathbf{b}_2,\mathbf{a}_1} \cap X_{\mathbf{b}_1,\mathbf{a}_1} = \emptyset$ we have

$$X_{\mathbf{b}_1,\mathbf{a}_1} \subset X_{\mathbf{a}_1,\mathbf{b}_2}.$$

Since $X_{\mathbf{a}_1,\mathbf{b}_2} \cap X_{\mathbf{a}_2,\mathbf{b}_2} = \emptyset$ we have

$$X_{\mathbf{a}_2,\mathbf{b}_2} \subset X_{\mathbf{b}_2,\mathbf{a}_1}.$$

Next we have $X_{\mathbf{b}_1,\mathbf{a}_2} \cup X_{\mathbf{a}_2,\mathbf{b}_1} = X$ and this union is disjoint. Since $X_{\mathbf{b}_1,\mathbf{a}_2} \cap X_{\mathbf{b}_2,\mathbf{a}_2} = \emptyset$ we have

$$X_{\mathbf{b}_2,\mathbf{a}_2} \subset X_{\mathbf{a}_2,\mathbf{b}_1}.$$

Since $X_{\mathbf{a}_2,\mathbf{b}_1} \cap X_{\mathbf{a}_1,\mathbf{b}_1} = \emptyset$ we have

$$X_{\mathbf{a}_1,\mathbf{b}_1} \subset X_{\mathbf{b}_1,\mathbf{a}_2}.$$

Finally we have $X_{\mathbf{b}_2,\mathbf{a}_2} \cup X_{\mathbf{a}_2,\mathbf{b}_2} = X$. In the same way as above this shows

$$X_{\mathbf{b}_1,\mathbf{a}_2} \subset X_{\mathbf{a}_2,\mathbf{b}_2},$$

and

$$X_{\mathbf{a}_1,\mathbf{b}_2} \subset X_{\mathbf{b}_2,\mathbf{a}_2}.$$

Combining these eight inclusions we obtain the first two identities of this proposition.

The sets X_1 and X_2 are disjoint because $X_1 = X_{\mathbf{b}_1,\mathbf{a}_1}$ and $X_2 = X_{\mathbf{a}_1,\mathbf{b}_1}$ and by Proposition 5.3 we have that $X_{\mathbf{b}_1,\mathbf{a}_1}$ and $X_{\mathbf{a}_1,\mathbf{b}_1}$ are disjoint. □

5.2. **The proof of Theorem 1.1.** Let $\Theta < \mathrm{Homeo}(M)$ be the group generated by the elements from all four groups $\Gamma(\mathbf{a}_i, \mathbf{b}_j)$.

Definition 5.3. *Let*

$$O = \bigcup_{\widetilde{\theta} \in \Theta} \widetilde{\theta}(O_0).$$

We have

Proposition 5.4. *Let $\widehat{O}$ be the lift of O to P and let $\widehat{f} : \overline{P} \to \overline{P}$ be a lift of $\mathcal{E}(t_\gamma)$ to $\overline{P}$. Assume that the points p and q belong to the same connected component of the set $\widehat{O}$. Then the cyclic group generated by $\widehat{f}$ is K long range Lipschitz on the pair of points p, q, for some $K > 0$ (the constant K may depend on the choice of p and q)*

Proof. Every homeomorphism from Θ setwise preserves the minimal annulus $A_{\min}$. Let $\widetilde{\theta} \in \Theta$ and let $\widehat{\theta} : \overline{P} \to \overline{P}$ be a lift. Let $K(\widetilde{\theta})$ be defined so that for every two points $z_1, z_2 \in \overline{P}$, such that $|z_1 - z_2| \leq C_{\min}$, we have that $|\widehat{\theta}(z_1) - \widehat{\theta}(z_2)| < K(\widetilde{\theta})$. The constant $K(\widetilde{\theta})$ exists because $\widehat{\theta}$ is uniformly continuous on $\overline{P}$ (it commutes with the translation for 1), and $K(\widetilde{\theta})$ does not depend on the choice of a lift $\widehat{\theta}$.

Fix $\widetilde{\theta} \in \Theta$. Every point $r \in O_0$ is contained in the interior of the component $S_r(i,j) \in \mathbf{S}(i,j)$, for some pair i, j. A single lift of $S_r(i,j)$ to P has the Euclidean diameter less than $C_{\min}$. Let $S = \widetilde{\theta}(S_r(i,j))$ and denote by $\widehat{S}$ a single lift of S to P. Then the Euclidean diameter of $\widehat{S}$ is less than $K(\widetilde{\theta})$. The homeomorphism $\mathcal{E}(t_\gamma)$ commutes with $\widetilde{\theta}$ (also recall that $\mathcal{E}(t_\gamma)$ respects the decomposition $\mathbf{S}(i,j)$). This implies that the set $\widehat{f}^n(\widehat{S})$ is a single lift of the interior of some $\widetilde{\theta}(S_q(k,l))$. This shows that the Euclidean diameter of the set $\widehat{f}^n(\widehat{S})$ is less than $K(\widetilde{\theta})$.

Since p and q are in the same connected component of $\widehat{O}$ there exists an arc $l \subset \widehat{O}$ with the endpoints p and q. Every point on l belongs to a single lift of some component $\widetilde{\theta}(S_r(i,j))$, for some $\widetilde{\theta} \in \Theta$, some $r \in O_0$ and some pair i, j. Since l is a closed subset of $\widehat{O}$ we can find finitely many such components $\widetilde{\theta}(S_r(i,j))$. Therefore the arc l is covered by finitely many sets $D_1, ..., D_k \subset P$, where each D_t, $t = 1, .., k$, is a single lift of some $\widetilde{\theta}(S_r(i,j))$ to P. Therefore there exists a constant $K' > 0$ such that the Euclidean diameter of every set $\widehat{f}^n(D_i)$, $n \in \mathbf{N}$, is less than K'. This shows that $|\widehat{f}^n(p) - \widehat{f}^n(q)| \leq kK'$, for every $n \in \mathbf{N}$. Set $K = kK'$. This proves the proposition. □

Proposition 5.5. *We have $(A_{\min} \setminus O) \subset \mathcal{I}(3C_{\min}, \mathcal{E}(t_\gamma), A_{\min})$.*

Proof. If $p \in (A_{\min} \setminus O)$ then either $p \in E$ or $p \in X$ (in fact this is true by definition for every $p \in (A_{\min} \setminus O_0)$). Assume that $p \in E$. Then there exists a component $S_p(i,j)$ such that $\mathcal{E}(t_\gamma)(S_p(i,j)) = S_p(i,j)$. Let $\widehat{S}_p(i,j)$ be a single lift of $S_p(i,j)$ to P and let $\widehat{p} \in \widehat{S}_p(i,j)$ be the corresponding lift of p to P. Moreover let $\widehat{f}$ be the lift of $\mathcal{E}(t_\gamma)$ to P that setwise fixes $\widehat{S}_p(i,j)$. We have that the Euclidean diameter of $\widehat{S}_p(i,j)$ is less than $C_{\min}$ and $\widehat{f}(\widehat{S}_p(i,j)) = \widehat{S}_p(i,j)$. This shows that $|\widehat{f}^n(\widehat{p}) - \widehat{p}| \leq C_{\min}$, for every $n \in \mathbf{Z}$. Therefore $p \in \mathcal{I}(2C_{\min}, \mathcal{E}(t_\gamma), A_{\min})$.

Assume that $p \in X$. We may assume that $p \in X_1$ (the case $p \in X_2$ is treated in the same way). Let $S_p(\mathbf{a}_1) \in \mathbf{S}(\mathbf{a}_1)$ be the corresponding component that contains p. There are two cases to consider.

The first case is when $S_p(\mathbf{a}_1) \cap E \neq \emptyset$. Let $q \in S_p(\mathbf{a}_1) \cap E \neq \emptyset$. Let $S_q(*)$ be the component such that $\mathcal{E}(t_\gamma)(S_q(*)) = S_q(*)$. This implies that

$$\mathcal{E}(t_\gamma)^n(S_p(\mathbf{a}_1)) \cap S_q(*) \neq \emptyset,$$

for every $n \in \mathbf{Z}$. Let $\widehat{q}$ be a lift of q to P and denote by $\widehat{S}_q(*)$ the corresponding lift of $S_q(*)$ to P that contains $\widehat{q}$. Let $\widehat{S}_p(\mathbf{a}_1)$ be the corresponding lift of $S_p(\mathbf{a}_1)$ to P such that $\widehat{q} \in \widehat{S}_p(\mathbf{a}_1)$. Also let $\widehat{p}$ be the lift of p to P such that $\widehat{p} \in \widehat{S}_p(\mathbf{a}_1)$. Finally let $\widehat{f}$ be the lift of $\mathcal{E}(t_\gamma)$ to P that setwise fixes the set $\widehat{S}_q(*)$.

Then for every $n \in \mathbf{Z}$ we have that $\widehat{f}^n(\widehat{p})$ belongs to the set $\widehat{f}^n(\widehat{S}_p(\mathbf{a}_1))$, and $\widehat{f}^n(\widehat{S}_p(\mathbf{a}_1))$ has non-empty intersection with $\widehat{S}_q(*)$. Since the Euclidean diameter of each of the sets $\widehat{S}_q(i,j)$ and $\widehat{f}^n(\widehat{S}_p(\mathbf{a}_1))$, for every $n \in \mathbf{Z}$ is less than $C_{\min}$, we have that $|\widehat{f}^n(\widehat{p}) - \widehat{f}^m(\widehat{p})| \leq 3C_{\min}$, for every m, $n \in \mathbf{Z}$ (that is for every $m, n \in \mathbf{Z}$ we have that the points $\widehat{f}^n(\widehat{p})$ and $\widehat{f}^m(\widehat{p})$ are contained in the union of three sets, and each of these three sets has the Euclidean diameter at most $C_{\min}$). This shows that $p \in \mathcal{I}(3C_{\min}, \mathcal{E}(t_\gamma), A_{\min})$.

The second case is when $S_p(\mathbf{a}_1) \cap E = \emptyset$. Let $q \in S_p(\mathbf{a}_1)$ be any point such that q does not belong to O. Then either $q \in X_1$ or $q \in X_2$, and not both can hold since $X_1 \cap X_2 = \emptyset$. Since $p \in X_1$ we have that $\widetilde{b}_1(S_p(\mathbf{a}_1)) = S_p(\mathbf{a}_1)$, for every $\widetilde{b}_1 \in \Gamma(\mathbf{b}_1)$. If $q \in X_2$ then by Lemma 5.1 we would have $\widetilde{b}_1(S_q(\mathbf{a}_1)) \neq S_q(\mathbf{a}_1)$, for some $\widetilde{b}_1 \in \Gamma(\mathbf{b}_1)$. But $S_p(\mathbf{a}_1) = S_q(\mathbf{a}_1)$, so we find that $q \in X_1$. This implies that $\widetilde{a}_2(S_q(\mathbf{b}_1)) = S_q(\mathbf{b}_1)$, for every $\widetilde{a}_2 \in \Gamma(\mathbf{a}_2)$, and for every $q \in (S_p(\mathbf{a}_1) \setminus O)$.

Let $\widehat{p}$ be a lift of p to P and let $\widehat{S}_p(\mathbf{a}_1)$ be the single lift of $S_p(\mathbf{a}_1)$ to P such that $\widehat{p} \in \widehat{S}_p(\mathbf{a}_1)$. Let $\widehat{S}_p(\mathbf{b}_1)$ be the lift of $S_p(\mathbf{b}_1)$ to P that contains $\widehat{p}$. For $\widetilde{a}_1 \in \Gamma(\mathbf{a}_1)$ let $\widehat{a}_1$ be the lift to P that setwise fixes $\widehat{S}_p(\mathbf{a}_1)$. Also for every $\widetilde{b}_1 \in \Gamma(\mathbf{b}_1)$ let $\widehat{b}_1$ be the lift to P that setwise fixes $\widehat{S}_p(\mathbf{b}_1)$ (we do not need this fact, but it can be easily shown that the lift $\widehat{b}_1$ setwise fixes the set $\widehat{S}_p(\mathbf{a}_1)$ as well). For $\widetilde{a}_2 \in \Gamma(\mathbf{a}_2)$ let $\widehat{a}_2$ be the lift to P that setwise fixes $\widehat{S}_p(\mathbf{b}_1)$.

Remark. Observe that each $\widehat{a}_2$ and $\widehat{b}_1$ has a fixed point in P. This follows from the assumption that $\widehat{a}_2(\widehat{S}_p(\mathbf{b}_1)) = \widehat{b}_1(\widehat{S}_p(\mathbf{b}_1)) = \widehat{S}_p(\mathbf{b}_1)$, and since $\widehat{S}_p(\mathbf{b}_1)$ is an acyclic set we see that each $\widehat{a}_2$ and $\widehat{b}_1$ has a fixed point in $\widehat{S}_p(\mathbf{b}_1)$.

Next we want to show that every $\widehat{a}_2 \in \Gamma(\mathbf{a}_2)$ setwise preserves the lift of every component from $\mathbf{S}(\mathbf{b}_1)$ that intersects $\widehat{S}_p(\mathbf{a}_1)$ in some point that does not belong to $\widehat{O}$ (where $\widehat{O}$ is the total lift of O to P). Fix $\widehat{a}_2$. Let $\widehat{S}_q(\mathbf{b}_1)$ be the lift of a component from $\mathbf{S}(\mathbf{b}_1)$ such that $\widehat{S}_q(\mathbf{b}_1)$ intersects $\widehat{S}_p(\mathbf{a}_1)$, and such that this intersection contains a point $\widehat{q}$ that is not in $\widehat{O}$. Then $\widehat{q} \in \widehat{X}_1$, where $\widehat{X}_1$ is the total lift of X_1 to P. This yields the relation $\widehat{a}_2(\widehat{S}_q(\mathbf{b}_1)) = \widehat{S}_q(\mathbf{b}_1) + k$, for some $k \in \mathbf{Z}$. Let $\widehat{S}_q(2,1)$ be the lift of the corresponding component from $\mathbf{S}(2,1)$ that contains q. Since each $\widehat{a}_2$ and $\widehat{b}_1$ has a fixed point in P, from Proposition 4.8 we see that $\widehat{a}_2(\widehat{S}_q(2,1)) = \widehat{b}_1(\widehat{S}_q(2,1)) = \widehat{S}_q(2,1)$. But then $\widehat{S}_q(2,1)$ contains both $\widehat{S}_q(\mathbf{b}_1)$ and $\widehat{S}_q(\mathbf{b}_1) + k$. Since $S_q(2,1)$ is acyclic we see that $k = 0$ that is $\widehat{a}_2$ setwise fixes $\widehat{S}_q(\mathbf{b}_1)$.

Let $\Theta(\mathbf{a})$ be the group generated by the elements from $\Gamma(\mathbf{a}_1)$ and $\Gamma(\mathbf{a}_2)$. Let $\widetilde{\theta} \in \Theta(\mathbf{a})$. Then $\widetilde{\theta} = \widetilde{a}_1^1 \circ \widetilde{a}_2^1 \circ \widetilde{a}_1^2 \circ \widetilde{a}_2^2 \circ ... \circ \widetilde{a}_1^k \circ \widetilde{a}_2^k$, for some $\widetilde{a}_i^j \in \Gamma(\mathbf{a}_i)$, where $i = 1, 2$. By appropriately choosing the lifts of each $\widetilde{a}_i^j$ we see that there exists a lift $\widehat{\theta} : P \to P$ of $\widetilde{\theta}$ such that $\widehat{\theta}(\widehat{p})$ belongs to a lift of some component from $\mathbf{S}(\mathbf{b}_1)$ to P that intersects $\widehat{S}_p(\mathbf{a}_1)$, and this intersection contains a point that is not in $\widehat{O}$ (here we use that $\widetilde{\theta}(O) = O$). Since $\mathcal{E}(t_\gamma) \in \Theta(\mathbf{a})$ we have that $p \in \mathcal{I}(2C_{\min}, \mathcal{E}(t_\gamma), A_{\min})$ (that is for every $m, n \in \mathbf{Z}$ we have shown that the points $\widehat{f}^n(\widehat{p})$ and $\widehat{f}^m(\widehat{p})$ are contained in the union of two sets, and each of these two sets has the Euclidean diameter at most $C_{\min}$). □

Now we are ready to prove Theorem 1.1. Let $K = 3C_{\min}$. Let $D_{\min}$ be the corresponding connected component of the set $A_{\min} \setminus \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min})$ from Proposition 4.7. By the previous proposition we have that

$$D_{\min} \subset A_{\min} \setminus \mathcal{I}(K, \mathcal{E}(t_\gamma), A_{\min}) \subset O.$$

On the other hand by Proposition 5.4 we have that if $\widehat{f}$ denotes a lift of $\mathcal{E}(t_\gamma)$ to P, then the cyclic group genereted by $\widehat{f}$ is N long range Lipschitz on the pair of points p, q in $D_{\min}$, for some $N > 0$ (the constant N may depend on the choice of p and q). Therefore we may apply Lemma 3.1. This lemma implies that $\rho(\mathcal{E}(t_\gamma), A_{\min}) = 0$. But this is a contradiction.

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